INFINITE PRESENTABILITY OF GROUPS AND CONDENSATION

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ABSTRACT. We describe various classes of infinitely presented groups, in connection with the notion of minimal presentation, and of condensation in the space of marked groups. In particular, we prove that any infinitely presented metabelian group is a condensation group. In contrast, we construct various examples of finitely generated groups with no minimal presentation, including infinitely presented groups with Cantor-Bendixson rank 1.

1. INTRODUCTION

1.A. **Infinitely presented and infinitely minimally presented groups.** Recall that a group is finitely presented if it has a presentation by a finite generating set, subject to a finite set of relators. In this paper, we especially focus on the class of **infinitely presented groups**, which we define as finitely generated groups that are not finitely presented: these groups have a presentation over a finite generating subset S

$$\langle S \mid (r_k)_{k \geq 1} \rangle \tag{1.1}$$

such that no finite subset of the relators defines the same group, i.e. for every n, the natural epimorphism

$$\langle S \mid (r_k)_{1 \leq k \leq n} \rangle \twoheadrightarrow \langle S \mid (r_k)_{k \geq 1} \rangle$$

is non-injective. There are natural stronger variants of this condition. One is that the presentation be *minimal*, in the sense that for all n, the natural epimorphism

$$\langle S \mid (r_k)_{n \neq k \geq 1} \rangle \twoheadrightarrow \langle S \mid (r_k)_{k \geq 1} \rangle$$

is non-injective. If a group admits such a presentation, it is called *infinitely minimally presented*. The first purpose of the present work is to exhibit examples of infinitely presented groups that are not infinitely minimally presented, and thus have no minimal presentation over any finite generating set.

Theorem 1.1. *There exist finitely generated, infinitely presented groups that are not infinitely minimally presented.*

In the course of the paper, we will present various examples of infinitely presented groups admitting no minimal presentation. The simplest ones are central quotients of the famous Abels' finitely presented solvable groups, see Corollary 3.6. These examples actually lie outside of a wider class, namely the class of infinitely independently presented groups (INIP for short). More tractable than infinitely minimally presented groups, INIP groups allow us to derive new examples in the condensation part of the space of marked groups as well as several condensation criteria.

1.B. **Infinitely independently presented groups.** Let G be a group and $(N_i)_{i \in I}$ be a family of normal subgroups of G. For $J \subset I$, we define N_J as the subgroup generated by $\bigcup_{j \in J} N_j$. We say that the family (N_i) is *independent* if the map $J \mapsto N_J$ is injective. Given an epimorphism between groups $G \twoheadrightarrow H$ with kernel N, we say that H is *infinitely independently presented* (or *INIP* for short) *relative to* G if there exists an infinite independent family $(N_i)_{i \in I}$ of normal subgroups of G such that $N_I = N$. If H is INIP relative to G and G is finitely presented then H is INIP relative to any other finitely presented extension (Lemma-Definition 4.3); H is called an *INIP* group in this case. An infinitely minimally presented group yields an infinite independent family of relators in the free group and hence an infinite independent family of normal subgroups. Finitely generated infinitely minimally presented groups are therefore INIP. In addition, the following holds

Proposition 1.2 (Prop. 4.5)**.** *The existence of a minimal presentation, for a finitely generated group, does not depend on the choice of a finite generating family.*

Based on a delicate construction by Ould Houcine, our Example 4.23 shows that the class of INIP groups is strictly larger than the class of finitely generated infinitely minimally presented groups. Conditions under which a wreath product, a wreathed Coxeter group or a small cancellation group is INIP are investigated in Section 4.B. This section also provides us with a general criterion based on the Schur multiplier:

Proposition 1.3 (Prop. 4.19)**.** *Let G be a finitely generated group and assume that the Schur multiplier $H_2(G, \mathbf{Z})$ is not minimax. Then G is infinitely independently presented.*

1.C. **The space $\mathcal{G}_m$ of marked groups.** Let m be a positive integer. A *group marked by the index set $I_m = \{1, 2, \ldots, m\}$* is a group G, together with a map $\iota_G \colon I_m \to G$ whose image generates G. Two marked groups (G, ι_G) and (H, ι_H) are said to be isomorphic if there is an isomorphism $\varphi \colon G \xrightarrow{\sim} H$ satisfying the relation $\iota_H = \varphi \circ \iota_G$.

We denote by $\mathcal{G}_m$ the set of groups marked by I_m, up to isomorphism of marked groups.

It is useful to have an explicit model of the set $\mathcal{G}_m$. To that end, we introduce the free group F_m with basis I_m and the marked group (F_m, ι_{F_m}) where $\iota_{F_m} \colon I_m \hookrightarrow F_m$ denotes the obvious embedding.

Given a group (G, ι_G) marked by I_m there exists then a unique epimorphism ρ of the free group F_m onto G satisfying the relation $\rho \circ \iota_{F_m} = \iota_G$. The kernel R of ρ does not change if the marked group (G, ι_G) is replaced by an isomorphic copy. The set $\mathcal{G}_m$ is thus parameterized by the normal subgroups of F_m.

The space $\mathcal{G}_m$ carries a natural topology, which was introduced by Chabauty [Cha50] (see [Gri05], [CG05, Sec. 2.2] or [CGP07, p. 258] for more details), which has a convenient description when $\mathcal{G}_m$ is identified with the set of normal subgroups of F_m. Then the collection of all sets of the form

$$\mathcal{O}_{\mathcal{F}, \mathcal{F}'} = \{S \lhd F_m \mid \mathcal{F} \subseteq S \text{ and } S \cap \mathcal{F}' = \emptyset\}, \tag{1.2}$$

where $\mathcal{F}$ and $\mathcal{F}'$ range over *finite* subsets of F_m, is a basis of the topology on $\mathcal{G}_m$.

1.D. **Cantor-Bendixson rank of a finitely generated group.** If T is a topological space, its subset of accumulation points gives rise to a subspace T'. We now apply this construction to the topological space $X = \mathcal{G}_m$ and iterate it. The outcome is a family of subspaces; it starts with

$$X^{(0)} = X, \quad X^{(1)} = X', \ldots, X^{(n+1)} = X^{(n)'}, \ldots, X^{(\omega)} = \bigcap_n X^{(n)},$$

and is continued by transfinite induction. This family induces a function on $\mathcal{G}_m$ with ordinal values, called *Cantor-Bendixson rank* or CB-rank for short (see §2.A for a more detailed description).

The points with CB-rank 0 are nothing but the isolated points of X; the points with CB-rank 1 are the points which are not isolated, but which can be approximated by isolated points and only by them. At the other extreme, are the points which lie in the intersection of all $X^{(\alpha)}$; they are called *condensation points*.

A finitely generated group G is said to have CB-rank α if it occurs as a point of CB-rank α in a space $\mathcal{G}_m$ for some integer m. It is a remarkable fact that this rank does not depend on the generating system $\iota\colon I_m \to G$ and, in particular, not on m (see part (3) of [CGP07, Lem. 1.3]). The CB-rank is thus a group theoretic property; it allows to talk about "isolated groups" and "condensation groups". A sufficient condition for a finitely generated group Q to be both condensation and infinitely presented is to be of *extrinsic condensation*, in the sense that for every finitely presented group G and epimorphism $G \twoheadrightarrow Q$, the kernel K has uncountably many subgroups that are normal in G.

1.E. **Condensation groups.** One goal of this paper is to exhibit various classes of condensation groups. To put these classes into perspective, we begin with a few words on groups of Cantor-Bendixson rank 0 or 1.

1.E.1. *Isolated groups and groups of CB-rank 1.* Isolated groups can be characterized in a simple manner. The following result is a restatement of [Man82, Prop. 2(a)]. Its topological interpretation was obtained independently in [Gri05, Thm. 2.1] and [CGP07, Prop. 2.2].

Proposition 1.4. *A finitely generated group is isolated if, and only if, it admits a finite presentation and has a finite collection (M_j) of nontrivial normal subgroups such that every nontrivial normal subgroup contains one of them.*

Obvious examples of isolated groups are therefore finite groups and finitely presentable simple groups. But there exists many other isolated groups (see [CGP07, Sec. 5]).

Proposition 1.4 contains the observation that every isolated group admits a finite presentation. Actually, it is easier to detect finitely presented groups that are non-condensation than infinitely presented groups with this property. Indeed, if a group G comes with an epimorphism $\rho\colon F_m \twoheadrightarrow G$, and if the kernel R of ρ is the normal closure of a finite set, $\mathcal{R}$ say, then

$$\mathcal{O}_{\mathcal{R},\emptyset} = \{S \lhd F_m \mid R \subseteq S\}$$

is a neighbourhood of G. Thus, if G is finitely presented, then a basis of neighbourhoods of G can be described as

$$\mathcal{O}_{\mathcal{F}'}^G = \{S \lhd G \mid S \cap \mathcal{F}' = \emptyset\}, \tag{1.3}$$

where $\mathcal{F}'$ ranges over finite subsets of $G \smallsetminus \{1\}$. The latter description does not refer to any free group mapping onto G.

The previous remark implies that an infinite cyclic group and, more generally, a finitely presented, infinite residually finite, just-infinite group has CB-rank 1. A simple instance [McC68] of such a group is consider, for any $n \geq 2$, the semidirect product $\mathbf{Z}_0^n \rtimes \mathrm{Sym}(n)$, where $\mathbf{Z}_0^n$ is the subgroup of $\mathbf{Z}^n$ of n-tuples with zero sum, and $\mathrm{Sym}(n)$ is the symmetric group, acting on $\mathbf{Z}^n$ by permutation of coordinates. A more elaborate example, due to Mennicke [Men65] is $\mathrm{PSL}_n(\mathbf{Z})$ if $n \geq 3$. By contrast, most infinitely presented groups are known or expected to be condensation and in particular no well-known infinitely presented group seems to have CB-rank 1; but such groups exist as can be seen from

Theorem 1.5 (Thm. 3.9)**.** *There exist infinitely presented groups with Cantor-Bendixson rank 1. Moreover, they can be chosen to be nilpotent-by-abelian.*

1.E.2. *Criteria.* It is an elementary fact that INIP groups are infinitely presented extrinsic condensation groups (Remark 5.3), so that all INIP criteria in Section 4 are also condensation criteria. Four other criteria for condensation groups will be established. Our first main criterion covers both finitely presented and infinitely presented groups:

Proposition 1.6 (Cor. 5.4)**.** *Every finitely generated group with a normal non-abelian free subgroup is a condensation group.*

The proposition is actually an elementary consequence of the following difficult result independently due to Adian, Olshanskii, and Vaughan-Lee (see §4.C): any non-abelian free group has uncountably many characteristic subgroups. As a simple application, the non-solvable Baumslag-Solitar groups

$$\mathrm{BS}(m,n) = \langle t, x \mid tx^m t^{-1} = x^n \rangle \qquad (|m|, |n| \geq 2)$$

are condensation groups.

Theorem 1.7 (Cor. 6.3)**.** *Let G be a finitely generated group with an epimorphism $\pi : G \twoheadrightarrow \mathbf{Z}$. Suppose that π does not split over a finitely generated subgroup of G, i.e. there is no decomposition of G as an HNN-extension over a finitely generated subgroup for which the associated epimorphism to $\mathbf{Z}$ is equal to π. Then G is an extrinsic condensation group (and is thus infinitely presented).*

Theorem 1.7 is based on Theorem 6.1 which is an extension of [BS78, Thm. A]; the latter states that every homomorphism from a finitely presented group onto the group $\mathbf{Z}$ splits over a finitely generated subgroup.

1.E.3. *Link with the geometric invariant.* If G is not isomorphic to a non-ascending HNN-extension over any finitely generated subgroup, Theorem 1.7 can be conveniently restated in terms of the *geometric invariant* $\Sigma(G) = \Sigma_{G'}(G)$ introduced in [BNS87]. We shall view this invariant as an open subset of the real vector space $\mathrm{Hom}(G, \mathbf{R})$ (see §6.B for more details). Writing $\Sigma^c(G)$ for the complement of $\Sigma(G)$ in $\mathrm{Hom}(G, \mathbf{R})$, we have

Corollary 1.8 (Cor. 6.10)**.** *Let G be a finitely generated group and assume that G is not isomorphic to a non-ascending HNN-extension over any finitely generated subgroup (e.g., G has no non-abelian free subgroup). If $\Sigma^c(G)$ contains a rational line then G is an extrinsic condensation group (and is thus infinitely presented).*

This corollary applies to several classes of groups G where $\Sigma^c(G)$ is known, such as finitely generated metabelian groups or groups of piecewise linear homeomorphisms.

The Cantor-Bendixson rank of finitely presented metabelian groups was computed in [Cor11a]. This corollary, which answers in particular [Cor11a, Qst. 1.7], completes the computation of the Cantor-Bendixson rank of metabelian groups.

Corollary 1.9 (Cor. 6.15)**.** *A finitely generated centre-by-metabelian group is a condensation group if, and only if, it is infinitely presented.*

The second principal application of Corollary 1.8 deals with groups of piecewise linear homeomorphisms of the real line. For some of these groups, the Σ-invariant has been computed in [BNS87, Thm. 8.1]. These computations apply, in particular, to Thompson's group F, the group consisting of all increasing piecewise linear homeomorphisms of the unit interval whose points of non-differentiability are dyadic rational numbers and whose slopes are integer powers of 2. We denote by χ_0 (resp. by χ_1) the homomorphism from F to $\mathbf{R}$ which maps f to the binary logarithm of its slope at 0 (resp. at 1). The homomorphisms χ_0 and χ_1 form a basis of $\mathrm{Hom}(F, \mathbf{Z})$; so every normal subgroup of F with infinite cyclic quotient is the kernel $N_{p,q} = \ker(p\chi_0 - q\chi_1)$, with $((p,q) \in \mathbf{Z}^2 \smallsetminus \{(0,0)\}$.

The discussion in [BNS87] describes which of the groups $N_{p,q}$ are finitely generated, or finitely presented, and how their invariants $\Sigma(N_{p,q})$ look like. Using Corollary 1.8 and methods from [CFP96], one can describe how the groups $N_{p,q}$ sit inside the space of marked groups.

Corollary 1.10. *(i) If $p \cdot q = 0$, then $N_{p,q}$ is infinitely generated,*
(ii) if $p \cdot q > 0$, then $N_{p,q}$ is finitely presented and isolated,
(iii) if $p \cdot q < 0$, then $N_{p,q}$ is a finitely generated extrinsic condensation group (and thus infinitely presented).

Here is a third application, where the group contains non-abelian free subgroups.

Corollary 1.11 (Ex. 6.7). *Let $A = \mathbf{Z}[x_n : n \in \mathbf{Z}]$ be the polynomial ring in infinitely many variables. Consider the ring automorphism ϕ defined by $\phi(x_n) = x_{n+1}$, which induces a group automorphism of the matrix group $\mathrm{SL}_d(A)$. Then for all $d \geq 3$, the semidirect product $\mathrm{SL}_d(A) \rtimes_\phi \mathbf{Z}$ is finitely generated and of extrinsic condensation (and thus infinitely presented).*

This leaves some open questions. Notably

Problem 1.12. Is any infinitely presented, finitely generated metabelian group infinitely independently presented (see Definition 4.1)? Does it admit a minimal presentation? Here are two test-cases

- The group $\mathbf{Z}[1/p]^2 \rtimes \mathbf{Z}$, with action by the diagonal matrix (p, p^{-1}). Here p is a fixed prime number.
- The group $\mathbf{Z}[1/pq] \rtimes \mathbf{Z}$, with action by multiplication by p/q. Here p, q are distinct prime numbers.
- The free metabelian group on $n \geq 2$ generators.

Contents

2. Preliminaries

2.A. **Elements of the Cantor-Bendixson theory.** Let X be a topological space. Its derived subspace $X^{(1)}$ is, by definition, the set of its accumulation points. Iterating over ordinals

$$X^{(0)} = X, \ X^{(\alpha+1)} = (X^{(\alpha)})^{(1)}, \ X^{(\lambda)} = \bigcap_{\beta < \lambda} X^{(\beta)} \text{ for } \lambda \text{ a limit ordinal,}$$

one constructs a non-increasing family $X^{(\alpha)}$ of closed subsets. If $x \in X$, we write

$$\mathrm{CB}_X(x) = \sup\{\alpha \mid x \in X^{(\alpha)}\}$$

if this supremum exists, in which case it is a maximum. Otherwise we write $\mathrm{CB}_X(x) = \mathfrak{C}$, where the symbol $\mathfrak{C}$ is not an ordinal.

We call $\mathrm{CB}_X(x)$ the *Cantor-Bendixson rank of x*. If $\mathrm{CB}_X(x) \neq \mathfrak{C}$ for all $x \in X$, i.e. if $X^{(\alpha)}$ is empty for some ordinal, we say that X is *scattered*. A topological space X is called *perfect* if it has no isolated point, i.e., $X^{(1)} = X$. As a union of perfect subsets is perfect, every topological space has a unique largest perfect subset, called its *condensation part* and denoted $\mathrm{Cond}(X)$. Clearly, $\mathrm{Cond}(X)$ is empty if and only if X is scattered and we have

$$\mathrm{Cond}(X) = \{x \in X \mid \mathrm{CB}_X(x) = \mathfrak{C}\}.$$

The subset $X \setminus \mathrm{Cond}(X)$ is the largest scattered open subset, and is called the *scattered part* of X.

In the sequel, we apply the concepts recalled in the above, either to the space $\mathcal{N}(G)$ of all normal subgroups of a finitely generated group G, or to the space of marked groups $\mathcal{G}_m$ (defined in Section 1.C). Both spaces have three crucial properties: they are *compact* (by Tychonoff's theorem); they are *metrizable*, for G and F_m are finitely generated, and hence countable, groups; and they are totally disconnected. It follows from the first two properties that $\mathcal{N}(G)$ and $\mathcal{G}_m$ have countable bases, and hence countable scattered parts. The three properties, when combined together with perfectness, characterize Cantor spaces. As the cardinality of $\mathcal{G}_m$ is $2^{\aleph_0}$ for $m > 1$ ([Neu37, Thm. 14], cf. [Hal54, Thm. 7]), the condensation part of of $\mathcal{G}_m$ is homeomorphic to the Cantor set (see [Gri85] for an explicit embedding of the Cantor space into $\mathcal{G}_m$). This last fact entails, in particular, that every neighbourhood of a condensation point in $\mathcal{G}_m$ contains $2^{\aleph_0}$ points.

2.B. **The Chabauty topology.** Let X be a topological space, and $F(X)$ the set of closed subsets in X. If $(U_j)_{j \in J}$ is a finite family of open subsets of X, and K is a compact subset in X, define

$$\mathcal{U}((U_j)_{j \in J}, K) = \{M \in F(X) \mid M \cap K = \emptyset; \ \ M \cap U_j \neq \emptyset, \forall j\}.$$

Then the $\mathcal{U}((U_j)_{j \in J}, K)$, when $(U_j)_{j \in J}$ ranges over all finite families of open subsets of X, and K ranges over compact subsets of X, form a basis of a topology, called Chabauty topology on X. (It is enough to consider the U_j ranging over a gives basis of open sets of X.)

When X is discrete, a basis of open sets is given by singletons. If (U_j) is a finite family of open sets, its union $\mathcal{F}$ is finite and the condition $\forall j, U_j \cap M \neq \emptyset$ simply means $\mathcal{F} \subset M$. Thus a basis of the Chabauty topology of $F(X) = 2^X$ is given by the

$$\mathcal{U}(\mathcal{F}, \mathcal{F}') = \{M \subset X \mid \mathcal{F} \subset M, \ M \cap \mathcal{F}' = \emptyset\},$$

which looks like the way we introduce it in §1.C.

If X is a locally compact (Hausdorff) space, the space $F(X)$ is Hausdorff and compact.

When $X = G$ is a locally compact group, an elementary verification shows that the set $\mathcal{N}(G)$ of closed normal subgroups is closed in $F(X)$.

Here we only address the case of discrete groups, where the above verifications are even much easier. In this context, it was observed by Champetier-Guirardel [CG05, §2.2] that the topology on the set of marked groups on d generators, as introduced by Grigorchuk [Gri85], coincides with the Chabauty topology on the set of normal subgroups of the free group on d generators.

3. Abels' groups and a group with Cantor-Bendixson rank 1

In this section, we prove Theorem 1.5.

3.A. **Abels' finitely presented matrix groups.** Our construction of an infinitely presented group with Cantor-Bendixson rank 1 is based on a sequence of groups studied by H. Abels and K. S. Brown ([Abe79] and [AB87]). Let $n \geq 3$ be a natural number, p be a prime number and $\mathbf{Z}[1/p]$ the ring of rationals with denominator a power of p. Define $A_n \leq \mathrm{GL}_n(\mathbf{Z}[1/p])$ to be the group of matrices of the form

$$\begin{pmatrix} 1 & \star & \cdots & \star & \star \\ 0 & \star & \cdots & \star & \star \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & \star & \star \\ 0 & 0 & \cdots & 0 & 1 \end{pmatrix}$$

with integral powers of p in the diagonal. The group A_n is finitely generated for $n \geq 3$. The group A_3, which was introduced by P. Hall in [Hal61], is infinitely presented; for $n \geq 4$, however, the group A_n has a finite presentation. Its centre $\mathcal{Z}(A_n)$ is isomorphic to the additive group of $\mathbf{Z}[1/p]$ and thus infinitely generated, whence the quotient group $A_n/\mathcal{Z}(A_n)$ is infinitely presented for all $n \geq 3$.

3.A.1. *Normal subgroups in the groups A_n.* We want to show next that the Cantor-Bendixson rank of the quotient $A_n/\mathcal{Z}(A_n)$ is countable for $n \geq 4$. As a preliminary step, we investigate the normal subgroups of $A_n/\mathcal{Z}(A_n)$. For this we need further definitions.

Definition 3.1. Let Ω be a group. A group G endowed with a left action of Ω by group automorphisms is called an Ω-*group*; an Ω-invariant subgroup H of G is called an Ω-*subgroup* of G. If H is a normal subgroup of G then G/H is an Ω-group for the induced Ω-action.

An Ω-group G satisfies *max-Ω* if every non-descending chain of Ω-subgroups of G stabilizes, or equivalently, if every Ω-subgroup of G is finitely generated as an Ω-group. If Ω is the group of inner automorphisms of G, the Ω-subgroups of G are the normal subgroups of G. If G satisfies max-Ω in this case, we say that G satisfies *max-n*.

We are now ready for the analysis of the normal subgroups of A_n with $n \geq 3$. Since its centre $\mathcal{Z}(A_n) \cong \mathbf{Z}[1/p]$ is infinitely generated, the group A_n does not satisfy max-n. Still, we have

Lemma 3.2. *The group $A_n/\mathcal{Z}(A_n)$ satisfies max-n.*

Proof. We denote by $\mathbb{1}_n$ the n-by-n identity matrix and by E_{ij} the n-by-n elementary matrix whose only non-zero entry is at position (i,j) with value 1; here $1 \leq i < j \leq n$. Set $U_{ij} = \mathbb{1}_n + \mathbf{Z}[1/p]E_{ij}$.

Let $D \simeq \mathbf{Z}^{n-2}$ be the subgroup of diagonal matrices in A_n. It is enough to check that $A_n/\mathcal{Z}(A_n)$ satisfies max-D. Since the max-D property is stable under extensions of D-groups (see the simple argument in [Rob96, 3.7]), it suffices to find a subnormal series of A_n whose successive subfactors satisfy max-D. This is easy: the group D and the unipotent one-parameter subgroups U_{ij} for $i < j$ are isomorphic to factors of a subnormal series of A_n and these factors satisfy max-D, except for the central subgroup U_{1n}. (Note that the additive group of U_{ij} is isomorphic to $\mathbf{Z}[1/p]$, and that some element of D acts on it by multiplication by p if $(i,j) \neq (1,n)$.) $\square$

We now proceed to count subgroups of Abels' groups. We begin by a general lemma, which will also be used in Example 5.15.

Lemma 3.3. *Let G be a countable group, in an extension $N \rightarrowtail G \twoheadrightarrow Q$. Suppose that N has at most countably many G-subgroups (that is, subgroups that are normal in G), and that Q satisfies max-n. Then G has at most countably many normal subgroups.*

Proof. Because of the assumption on N, it is enough to show that for every normal subgroup M_0 of G contained in N, there are at most countably many normal subgroups M of G such that $M \cap N = M_0$. Indeed, since M/M_0 is isomorphic, as a G-group, to a subgroup of Q, we see that it is finitely generated qua normal subgroup of M/M_0; in other words, M is generated, as a normal subgroup, by the union of M_0 with a finite set. Thus there are at most countably many possibilities for M. $\qquad\square$

Proposition 3.4. [Lyu84, Thm. 1] *The group A_n has exactly countably many normal subgroups, although it does not satisfy max-n.*

Proof. The centre $\mathcal{Z}(A_n)$ is isomorphic to $\mathbf{Z}[1/p]$ which is not finitely generated; therefore A_n cannot satisfies max-n. As $\mathcal{Z}(A_n) \simeq \mathbf{Z}[1/p]$ has only countably many subgroups, and $A_n/\mathcal{Z}(A_n)$ satisfies max-n, Lemma 3.3 shows that A_n has at most countably many normal subgroups. $\qquad\square$

Proposition 3.4 was only stated for $n = 4$ in [Lyu84] but the proof carries over the general case with no significant modification. However, we need a more precise description of the set of normal subgroups of A_n, which readily implies Proposition 3.4.

Lemma 3.5. *Let N be a normal subgroup of A_n. Then either*

- $N \subset \mathcal{Z}(A_n)$, *or*
- N *is finitely generated as a normal subgroup, and contains a finite index subgroup of $\mathcal{Z}(A_n)$.*

We denote by $U(A_n)$ the subgroup of unipotent matrices of A_n.

Proof. Suppose that N is not contained in $\mathcal{Z}(A_n)$. Set $M = N \cap U(A_n)$. We first prove that $Z' = M \cap \mathcal{Z}(A_n)$ has finite index in $\mathcal{Z}(A_n)$.

The image of N inside $A_n/\mathcal{Z}(A_n)$ cannot intersect trivially $K = U(A_n)/\mathcal{Z}(A_n)$ as it is a non-trivial normal subgroup and since K contains its own centralizer. Consequently M is not contained in $\mathcal{Z}(A_n)$. Since K is a non-trivial nilpotent group, the image $\bar{M}$ of M in K intersects $\mathcal{Z}(K)$ non-trivially (if an element of $\bar{M}$ is not central, perform the commutator with an element of K to get another non-trivial element of $\bar{M}$, and reiterate until we obtain a central element: the process stops by nilpotency). Thus M contains a matrix m of the form $\mathbf{1}_n + r_1 E_{1,n-1} + r_2 E_{2,n} + c E_{1,n}$ where one of the r_i is not zero. Taking the commutators of m with $U_{n-1,n}$ (resp. $U_{1,2}$) if $r_1 \neq 0$ (resp. $r_2 \neq 0$), we obtain a finite index subgroup Z' of $\mathcal{Z}(A_n) = U_{1,n}$ which lies in M. The proof of the claim is then complete.

Now A_n/Z' satisfies max-n by Lemma 3.2, and therefore the image of N in A_n/Z' is finitely generated as a normal subgroup. Lift finitely many generators to elements generating a finitely generated normal subgroup N' of A_n contained in N. As N is not contained in $\mathcal{Z}(A_n)$, N' cannot be contained in $\mathcal{Z}(A_n)$. The claim above, applied to N', shows that N' contains a finite index subgroup of $\mathcal{Z}(A_n)$. As the index of N' in N coincides with the index of $N' \cap Z'$ in Z', the former is finite. Therefore N is finitely generated as a normal subgroup. $\qquad\square$

3.A.2. *Cantor-Bendixson rank of the groups $A_n/\mathcal{Z}(A_n)$.* We get in turn

Corollary 3.6. *For every $n \geq 4$ the quotient group of A_n by its centre is an infinitely presented group with at most countable Cantor-Bendixson rank. In particular (anticipating on Lemma 5.2(i)), it is not INIP and admits no minimal presentation over any finite generating set.*

Proof. Since A_n is finitely presented [AB87] ([Abe79] for $n = 4$), there is an open neighborhood of $A_n/\mathcal{Z}(A_n)$ in the space of finitely generated groups which consists of marked quotients of A_n. This neighborhood is countable by Proposition 3.4 and so $A_n/\mathcal{Z}(A_n)$ is not a condensation group. But if so, its Cantor-Bendixson rank must be a countable ordinal (cf. the last part of §2.A). □

Remark 3.7. The Cantor-Bendixson rank of $A_n/\mathcal{Z}(A_n)$ for $n \geq 4$ can be computed explicitly: it is $n(n+1)/2 - 3$ and so coincides with the number of relevant coefficients in a "matrix" in A_n, viewed modulo $\mathcal{Z}(A_n)$. For $n = 4$, this number is 7. Similarly, the Cantor-Bendixson rank of A_n is $n(n+1)/2 - 2$ for $n \geq 4$. These claims can be justified as in [Cor11a, Lem. 3.19]. We shall not give any details of the verification; since instead we refine the construction in the next paragraph to get an example of an infinitely presented solvable group with Cantor-Bendixson rank one.

3.B. **Construction of a group with Cantor-Bendixson rank 1.** We begin by explaining informally the strategy of our construction. As we saw, if $Z \simeq \mathbf{Z}[1/p]$ is the centre of A_n and $n \geq 4$, then A_n/Z is an infinitely presented group and is not condensation; this is based on the property that Z, albeit infinitely generated, has, in a certain sense few subgroups. However, A_n/Z is not of Cantor-Bendixson rank one, notably because it has too many quotients.

We are going to construct a similar example, but using an artifact to obtain a group that cannot be approximated by its own proper quotients. We will start with a group A, very similar to A_5 but whose centre Z is isomorphic to $\mathbf{Z}[1/p]^2$, and $Z_0 \simeq \mathbf{Z}^2$ a free abelian subgroup of rank two.

We will consider a certain, carefully chosen subgroup V_1 of Z, containing Z_0, such that Z/V_1 and V_1/Z_0 are both isomorphic to $\mathbf{Z}(p^\infty)$. The factor group A/V_1 is infinitely presented and the quotient groups of the finitely presented group A/Z_0 make up an open neighbourhood $\mathcal{V}_0$ of A/V_1 in the space of marked groups. If there existed a smaller neighbourhood $\mathcal{V}$ of V_1 containing only subgroups of Z/Z_0 and in which V_1/Z_0 were the only infinite subgroup then A/V_1 would have have Cantor-Bendixson rank 1. At this point this seems illusory, because it can be checked elementarily that V_1 is a condensation point in the space of subgroups of Z. To remedy these shortcomings, we work in a semidirect product $A \rtimes C$, where C is the cyclic subgroup generated by some automorphism normalizing V_1. The point is that the construction will ensure that only few of the subgroups of Z will be normalized by C.

Both C and V_1 have to be chosen with care. We will prescribe the automorphism generating C to act on $Z \simeq \mathbf{Z}[1/p]^2$ in such a way that $\mathbf{Z}^2$ is invariant and that the action is diagonalizable over $\mathbf{Q}_p$ but not over $\mathbf{Q}$. If $\mathcal{D}_1$ is an eigenline in $\mathbf{Q}_p^2$, the subgroup V_1 will be defined as $(\mathcal{D}_1 + \mathbf{Z}_p^2) \cap \mathbf{Z}[1/p]^2$. The assumption of non-diagonalizability over $\mathbf{Q}$ will intervene when we need to ensure that C normalizes few enough subgroups in A.

3.B.1. *Choice of the group A.* Let p be a fixed prime number and let A be the subgroup of $A_5 \leq \mathrm{GL}_5(\mathbf{Z}[1/p])$ consisting of all matrices of the form

$$\begin{pmatrix} 1 & \star & \star & \star & \star \\ 0 & \star & \star & \star & \star \\ 0 & 0 & \star & \star & \star \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

The group A is an S-arithmetic subgroup of $\mathrm{GL}_5(\mathbf{Q})$ and so the question whether it admits a finite presentation can, by the theory developed by H. Abels in [Abe87], be reduced to an explicit computation of homology, which in the present case is performed in [Cor11b, §2.2]:

Lemma 3.8. *The group A is finitely presented.*

3.B.2. *Choice of the automorphism and the subgroup E.* The automorphism will be induced by conjugation by a matrix of the form

$$M_1 = \begin{pmatrix} \mathbb{1}_3 & 0 \\ 0 & M_0 \end{pmatrix} \in \mathrm{GL}_5(\mathbf{Z}),$$

where M_0 is any matrix in $\mathrm{GL}_2(\mathbf{Z})$ satisfying the following requirements:

(i) M_0 is not diagonalizable over $\mathbf{Q}$;

(ii) M_0 is diagonalizable over the p-adic numbers $\mathbf{Q}_p$.

An example of a matrix M_0 with properties (i) and (ii) is the companion matrix of the polynomial $X^2 + p^3 X - 1$: one verifies easily that (i) holds for every choice of p; to check (ii), one can use [Ser77, Sec. II.2] to prove that the root 1 of this polynomial in $\mathbf{Z}/p^3\mathbf{Z}$ lifts to a root in the ring of p-adic integers $\mathbf{Z}_p$. Note that M is allowed to be a matrix of finite order (thus of order 3, 4, or 6 by (i), the validity of (ii) depending on p).

Let λ_1 and λ_2 be the eigenvalues of the matrix M_0 in $\mathbf{Q}_p$. Since multiplication by M_0 induces an automorphism of $\mathbf{Z}_p^2$, these eigenvalues are units of the ring $\mathbf{Z}_p$; as M_0 cannot be a multiple of the identity matrix in view of requirements (i) and (ii), they are distinct. Let $\mathcal{D}_1$ and $\mathcal{D}_2$ denote the corresponding eigenlines in the vector space $\mathbf{Q}_p^2$. Then $\mathbf{Q}_p^2 = \mathcal{D}_1 \oplus \mathcal{D}_2$. By assumption, the intersection of $\mathcal{D}_i$ $(i = 1, 2)$ with the dense subgroup $\mathbf{Z}[1/p]^2$ are trivial so we rather consider the intersections

$$E_i = \mathbf{Z}[1/p]^2 \cap (\mathcal{D}_i + \mathbf{Z}_p^2).$$

Since $\mathcal{D}_i + \mathbf{Z}_p^2$ is an open subgroup and $\mathbf{Z}[1/p]$ is dense, we see that E_i is an extension of $\mathbf{Z}^2 = \mathbf{Z}_p^2 \cap \mathbf{Z}[1/p]^2$ by $(\mathcal{D}_i + \mathbf{Z}_p^2)/\mathbf{Z}_p^2 \simeq \mathbf{Z}(p^\infty)$, and $\mathbf{Z}[1/p]^2/E_i$ is isomorphic to $\mathbf{Q}_p^2/(\mathcal{D}_i + \mathbf{Z}_p^2) \simeq \mathbf{Z}(p^\infty)$.

Set $e_{ij}^a = \mathbb{1}_n + aE_{ij}$ for $a \in \mathbf{Z}[1/p]$, $1 \leq i \neq j \leq n$. Then the centre Z of A consists of the products $e_{14}^a \cdot e_{15}^b$. The map

$$\mu \colon \mathbf{Z}[1/p]^2 \xrightarrow{\sim} Z, \quad (a, b) \mapsto e_{14}^a \cdot e_{15}^b. \tag{3.1}$$

is an isomorphism and shows that Z is isomorphic to $\mathbf{Z}[1/p]^2$. Define $V_i = \mu(E_i)$. Note that $Z_0 \subset V_i \subset Z$ and both Z/V_i and V_i/Z_0 are isomorphic to $\mathbf{Z}(p^\infty)$. Notice that the group $C = \langle M_1 \rangle$ normalizes A, its centre Z as well as Z_0, V_1, and V_2. We are ready to define our group as

$$B = (A/V_1) \rtimes C. \tag{3.2}$$

If $\bar{V}_1$ denotes the image of V_1 in $\bar{A} = A/Z_0$, we see that $B = (\bar{A}/\bar{V}_1) \rtimes C$. Since $\bar{A}$ and hence $\bar{A} \rtimes C$ are finitely presented, the quotient groups of $\bar{A} \rtimes C$ form an open

neighbourhood of B in the space of marked groups. In what follows we need a smaller neighbourhood; to define it, we consider the (finite) set W_0 of elements of order p in $\bar{Z} - \bar{V}_1$ and $W = W_0 \cup \{w\}$, where w is a fixed nontrivial element of $\bar{V}_2 - \bar{V}_1$.

$$\mathcal{V} = \mathcal{O}_W^{\bar{A} \rtimes C} = \{N \lhd \bar{A} \rtimes C \mid N \cap W = \emptyset\}. \tag{3.3}$$

The set $\mathcal{V}$ is actually far smaller than its definition would suggest:

Theorem 3.9. *The set $\mathcal{V}$ consists only of $\bar{V}_1 \simeq \mathbf{Z}(p^\infty)$ and of finite subgroups of $\bar{Z}$, and so B is an infinitely presented, nilpotent-by-abelian group of Cantor-Bendixson rank one.*

3.C. **Proof of Theorem 3.9.** The proof is based on two lemmata. In each of them one of the defining properties of the matrix M_0 plays a crucial rôle. The first lemma exploits property (i) and reads:

Lemma 3.10. *Let N be a normal subgroup of A normalized by C and not contained in Z. Then N contains a finite index subgroup of Z. If N is in addition assumed to contain Z_0, then it contains Z.*

Proof. The proof is similar in many respects to that of Lemma 3.5, so we skip some details. We first claim that if N is any normal subgroup in A and $N \nsubseteq Z$, then N contains a nontrivial $\mathbf{Z}[1/p]$-submodule of $Z \simeq \mathbf{Z}[1/p]^2$.

Arguing as in the proof of Lemma 3.5, it follows that N contains an element in $[U(A), U(A)] - Z$, namely of the form

$$\begin{pmatrix} 1 & 0 & u_{13} & u_{14} & u_{15} \\ 0 & 1 & 0 & u_{24} & u_{25} \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} \quad \text{with} \quad (u_{13}, u_{24}, u_{25}) \neq (0, 0, 0).$$

If $u_{13} \neq 0$, then by taking commutators with elementary matrices in the places $(3, 4)$ and $(3, 5)$ we obtain a subgroup of Z of finite index. If $(u_{24}, u_{25}) \neq (0, 0)$, we take commutators with elementary matrices in the $(1, 2)$-place and find that N contains the image under μ of the $\mathbf{Z}[1/p]$-submodule generated by (u_{24}, u_{25}), so the claim is proved.

If $N \nsubseteq Z$ is assumed in addition to be normalized by C, which acts $\mathbf{Q}$-irreducibly by hypothesis (i), we deduce that $N \cap Z$ has finite index in Z in all cases. If $Z_0 \subset Z$, it follows that the image of $N \cap Z$ in Z/Z_0 has finite index; since Z/Z_0 has no proper subgroup of finite index, it follows that $Z \subset N$. $\qquad \square$

The second auxiliary result makes use of hypothesis (ii).

Lemma 3.11. *Let H be an M_0-invariant subgroup of $\mathbf{Z}[1/p]^2$. Suppose that H contains $\mathbf{Z}^2$ and $H/\mathbf{Z}^2$ is infinite. Then $H = \mathbf{Z}[1/p]^2$, or $H = p^{-n}\mathbf{Z}^2 + E_i$ for some $i \in \{1, 2\}$ and $n \geq 0$.*

Proof. We first claim that H contains either E_1 or E_2. Let H' be the closure of H in $\mathbf{Q}_p^2$. The assumption $H/\mathbf{Z}^2$ infinite implies that H' is not compact. Let us first check that H' contains either $\mathcal{D}_1$ or $\mathcal{D}_2$. We begin by recalling the elementary fact that any closed, p-divisible subgroup of $\mathbf{Q}_p$ is a $\mathbf{Q}_p$-linear subspace: indeed any closed subgroup is a $\mathbf{Z}_p$-submodule, which p-divisibility forces to be a $\mathbf{Q}_p$-submodule. Since H' is not compact, for any $n \geq 0$, $p^n H'$ has nontrivial intersection with the 1-sphere in $\mathbf{Q}_p^2$. Therefore, by compactness, $\bigcap_{n \geq 0} p^n H'$ is a nonzero M_0-invariant and p-divisible closed subgroup and is therefore a $\mathbf{Q}_p$-subspace of $\mathbf{Q}_p^2$, thus contains some $\mathcal{D}_i$.

Thus $\mathcal{D}_i + \mathbf{Z}_p \subset H'$. Let us check that $E_i \subset H$. Since E_i is contained in the closure H' of H, for all $x \in E_i$ there exists a sequence $x_n \in H$ such that x_n tends to x. But

$x_n - x \in \mathbf{Z}[1/p]^2$ and tends to zero in $\mathbf{Q}_p^2$, so eventually belongs to $\mathbf{Z}^2$, which is contained in H. So for large n, $x_n - x$ and hence x belongs to H and $E_i \subset H$, so the claim is proved.

If $H \neq \mathbf{Z}[1/p]^2$, consider the largest n such that $L = p^{-n}\mathbf{Z}^2 \subset H$. We wish to conclude that $H = L + E_i$. The inclusion $\supset$ is already granted. Observe that H/L contains a unique subgroup of order p. But an immediate verification shows that any subgroup of $\mathbf{Z}(p^\infty)^2$ strictly containing a subgroup isomorphic to $\mathbf{Z}(p^\infty)$, has to contain all the p-torsion. If H contains $L + E_i$ as a proper subgroup, we apply this to the inclusion of $(L + E_i)/L \simeq \mathbf{Z}(p^\infty)$ into H/L inside $\mathbf{Z}[1/p]^2/L \simeq \mathbf{Z}(p^\infty)^2$ and deduce that H/L contains the whole p-torsion of $\mathbf{Z}[1/p]^2/L$, so that H contains $p^{-(n+1)}\mathbf{Z}^2$, contradicting the maximality of n. So $H = L + E_i = p^{-n}\mathbf{Z}^2 + E_i$ and we are done.   □

Conclusion of the proof of Theorems 3.9 and 1.5. Clearly, as a quotient of the finitely generated group $\bar{A} \rtimes C$ by $\bar{V}_1 \simeq \mathbf{Z}(p^\infty)$, which is the increasing union of its finite characteristic subgroups and therefore is not finitely generated as a normal subgroup, the group B is not finitely presented.

Let N be a normal subgroup of $\bar{A} \rtimes C$ that lies in the neighbourhood $\mathcal{V}$ defined by equation (3.3). Then Lemma 3.10 (applied to the inverse image of $N \cap \bar{A}$ in $A \rtimes C$) implies that $N \cap \bar{A} \subset \bar{Z}$. Since $\bar{A}/\bar{Z} = A/Z$ contains its own centralizer in $A/Z \rtimes C$, it follows that $N \subset \bar{Z}$.

Assume now that N is infinite. By Lemma 3.11, we have $H = \bar{Z}$, or $H = (p^{-n}Z_0 + V_i)/Z_0$ for some $n \geq 0$ and $i \in \{1, 2\}$. But all these groups have nonempty intersection with W, except $\bar{V}_1$ itself. Thus the only infinite subgroup N in $\mathcal{V}$ is $\bar{V}_1$.

As finite subgroups of $\bar{Z}$ are isolated in the space of subgroups of $\bar{Z} \simeq \mathbf{Z}(p^\infty)^2$ by an easy argument (see for instance [CGP10, Prop. 2.1.1]), it follows that $\mathcal{V} - \{\bar{V}_1\}$ consists of isolated points; clearly $\bar{V}_1$ is not isolated as it can be described as the increasing union of its characteristic subgroups and it follows that $\bar{V}_1$ has Cantor-Bendixson rank one in $\mathcal{V}$, and therefore $\bar{V}_1$ has Cantor-Bendixson rank one in the space of normal subgroups of the finitely presented group $\bar{A} \rtimes C$, so $B = (\bar{A} \rtimes C)/\bar{V}_1$ has Cantor-Bendixson rank one.

To conclude the assertion in Theorem 1.5 that B is nilpotent-by-abelian, observe that the obvious decomposition $A = U(A) \rtimes D$ (where D is the set of diagonal matrices in A) extends to a semidirect decomposition $A \rtimes C = U(A) \rtimes (D \times C)$, which is nilpotent-by-abelian and admits B as a quotient.   □

Remarks 3.12. (a) The trivial subgroup of B is isolated in $\mathcal{N}(B)$ (i.e., B is finitely discriminable, see §6.D), for the above proof shows that any nontrivial normal subgroup of B contains the unique subgroup of order p in Z/V_1.

(b) If we require that modulo p, the matrix M_0 has two distinct eigenvalues, then Theorem 3.9 can be slightly improved: first it is enough to pick $W = \{w\}$, where w is an element of order p in $\bar{V}_2$, and the set $\mathcal{V}$ consists only of $\bar{V}_1 \simeq \mathbf{Z}(p^\infty)$ and its finite subgroups. Note that without this further assumption, it can happen that the intersection $\bar{V}_1 \cap \bar{V}_2$ be nontrivial, and even if this intersection is trivial, the finite subgroups in $\mathcal{V}$ are not necessarily all contained in $\bar{V}_1$.

4. Independent families and presentations

We introduce here the crucial notion of *independent* families of normal subgroups, which will play an essential role in the proof of Theorem 1.7. In view of the connection with the space of marked groups, we also introduce the notion of continuous independent families; in a first reading, the reader can focus on the statements involving independency and not care about continuous independency.

4.A. **Independent and minimal presentations.** Let G be a group and $(N_i)_{i \in I}$ be a family of normal subgroups of G (in the sequel, I will often be implicit). For $J \subset I$, we define N_J as the subgroup generated by $\bigcup_{j \in J} N_j$. The *standard map* associated to (N_i) is the map $\Phi = \Phi_{(N_i)_{i \in I}}$ from 2^I to $\mathcal{N}(G)$, mapping $J \subset I$ to N_J. Note that $N_\emptyset = \{1\}$.

Definition 4.1. We say that the family (N_i) is *independent* if Φ is injective, and *continuously independent* if Φ is injective and continuous (and thus a homeomorphic embedding).

Given an epimorphism between groups $G \twoheadrightarrow H$ with kernel N, we say that H is *infinitely independently presented* (*INIP* for short) *relative to G* if there exists an infinite independent family $(N_i)_{i \in I}$ of normal subgroups of G such that $\Phi(I) = N$.

Here is an elementary lemma describing how, practically, injectivity and continuity of the map Φ can be checked.

Lemma 4.2. *The standard map $\Phi : 2^I \to \mathcal{N}(G)$ associated to a sequence of normal subgroups $(N_i)_{i \in I}$ of a group G, is order-preserving. It is injective if and only if for every $i \in I$, the subgroup N_i is not included in $\Phi(I \smallsetminus \{i\})$, the normal subgroup generated by $\bigcup_{j \neq i} N_j$. If $J \subset I$, it is continuous at J if and only if for every $w \notin \Phi(J)$ there exists a cofinite subset P containing J such that $w \notin \Phi(P)$.*

Proof. It is trivial that Φ is order-preserving. The injectivity obviously implies the condition, since the latter can be restated as:

$$\Phi(I \smallsetminus \{i\}) \neq \Phi(I), \quad \forall i \in I;$$

conversely if this condition is satisfied, if J, K are distinct subsets, switching J and K if necessary we can suppose that some element $i \in J$ does not belong to K. So $N_i \subset \Phi(J)$ but since $\Phi(K) \subset \Phi(I \smallsetminus \{i\})$ we have $N_i \nsubseteq \Phi(K)$, so $\Phi(J) \neq \Phi(K)$.

For the continuity statement, let us first check that Φ is *lower semicontinuous* at every $J \subset I$. This means that given any finite subset F of $\Phi(J) = N_J$, there exists a neighbourhood $\mathcal{V}$ of J in 2^I such that for every $K \in \mathcal{V}$, the subgroup $\Phi(K) = N_K$ contains F. Now F is contained in the subgroup generated by $\bigcup_{j \in J} N_j$, so there exists a finite subset J_0 of J such that F is contained in N_{J_0}. Therefore if $\mathcal{V}$ is the set of subsets of I containing J_0, then it satisfies the above property and thus Φ is lower semicontinuous at J.

Thus Φ is continuous at $J \subset I$ if and only if Φ is *upper semicontinuous* at J, in the sense that for every finite subset F of $G \smallsetminus \Phi(J)$, there exists a neighbourhood $\mathcal{V}$ of J in 2^I such that $\Phi(K) \cap F = \emptyset$ for all $K \in \mathcal{V}$.

First assume that Φ is upper semicontinuous at J and assume that $w \notin \Phi(J) = N_J$. Then there exists a neighbourhood $\mathcal{V}$ of J such that for every $K \in \mathcal{V}$, we have $w \notin N_K$. Replacing $\mathcal{V}$ by a smaller neighbourhood if necessary, we can suppose that there exists finite subsets $J_0 \subset J$ and $W \subset I \smallsetminus J$ such that $\mathcal{V}$ is the set of subsets of I containing J_0 and having empty intersection with W. In particular, if P is the complement of W, then P is cofinite and $w \notin N_P$.

Conversely suppose that the condition is satisfied and let us show that Φ is upper semicontinuous at J. Let W be a finite subset of $G \smallsetminus \Phi(J)$. For every $w \in W$, there exists a cofinite subset $P_w \subset I$ containing J such that $w \notin \Phi(P_w)$. So if $P = \bigcap_w P_w$, then P is cofinite, contains J, and $N_P \cap W = \emptyset$. Since P is cofinite, the set 2^P of subsets of I contained in P is a neighbourhood of J in 2^I and for every $K \in 2^P$ we have $N_K \cap W = \emptyset$. This shows upper semicontinuity. $\square$

Note that if $(N_i)_{i \in I}$ is independent and $N_i \subset N$, then $\#I \leq \#N$. In particular, when dealing with countable groups, it will be enough to consider at most countable families. Also observe that if $(N_i)_{i \in I}$ is independent then every finite subfamily is independent;

conversely, if all finite subfamilies are independent, and if in addition all N_i are finitely generated qua normal subgroups, then (N_i) is independent.

The following elementary statement is an analogue to the fact that being finitely presented does not depend on the set of generators.

Lemma-Definition 4.3. *Let H be a finitely generated group. Then, given an epimorphism $G \twoheadrightarrow H$ with G finitely presented, the answer to whether H is INIP relative to G, does not depend on G, nor on the epimorphism $G \twoheadrightarrow H$; if this is true we say that H is* infinitely independently presented *(or INIP for short).*

Proof. It is enough to check that if $G_1 \twoheadrightarrow G_2 \twoheadrightarrow H$ with G_1, G_2 finitely presented, H is INIP relative to G_1 if and only if it is INIP relative to G_2.

Suppose that H is INIP relative to G_2, say by a sequence (M_n). Let (M'_n) be the inverse image of M_n in G_1. Then clearly H is INIP relative to G_1, by (M'_n).

Conversely, suppose that H is INIP relative to G_1, say by a sequence (M_n). Let W be a finite subset of G_1 generating, as a normal subgroup, the kernel N_0 of $G_1 \twoheadrightarrow G_2$. Then there exists n such that W is contained in $N = M_{\{0,\dots,n-1\}}$. Define T_k to be the subgroup of G_1 generated by N and M_k. The family $(T_k)_{k \geq n}$ is independent, because if $k \geq n$, we have $T_{\mathbf{N}_{\geq n} \smallsetminus \{k\}} = M_{\mathbf{N} \smallsetminus \{k\}}$ and thus $T_{\mathbf{N}_{\geq n} \smallsetminus \{k\}}$ is not equal to $T_{\mathbf{N}_{\geq n}} = M_{\mathbf{N}}$. It follows that $(T_k/N_0)_{k \geq n}$ is an independent sequence of normal subgroups of G_2, generating the kernel of $G_2 \twoheadrightarrow H$. $\qquad\square$

The notion of independent sequences was initially motivated by the following particular case.

Definition 4.4. *If (r_i) is a family of elements of a group G, we say that (r_i) is a (continuously) independent family of relators, if the family (N_i) is (continuously) independent, where N_i is the normal subgroup of G generated by r_i.*

We say that a group presentation $\langle S | \mathcal{R} \rangle$ is *minimal* if S is finite and no relator is redundant. In other words, this means that in the free group F_S, the family of normal subgroups generated by singletons in $\mathcal{R}$ is independent. If a group G admits such a presentation, it is called *minimally presented*.

Clearly every finitely presented group is minimally presented, since we can remove successively redundant relators. If an infinitely presented group admits a minimal presentation, then the family of relators has to be infinite countable and thus the group is INIP. Among the INIP groups provided in §4.B, several are actually minimally presented.

Proposition 4.5. *The existence of a minimal presentation, for a finitely generated group, does not depend on the choice of a finite generating family.*

We need the following trivial but useful lemma.

Lemma 4.6. *If $G = \langle S | \mathcal{R} \rangle$ is a presentation in which at most finitely many of the relators $r \in \mathcal{R}$ are redundant, then we can extract a minimal presentation $G = \langle S | \mathcal{R}_1 \rangle$, where $\mathcal{R}_1$ is obtained from $\mathcal{R}$ by removing only finitely many relators.*

Proof. Let F be the set of redundant relators. We remove redundant relators one by one until there are not any longer: at each step, the set of redundant relators is a subset of F and therefore the process stops after at most $|F|$ steps. $\qquad\square$

Proof of Proposition 4.5. Obviously, given two finite generating families S and T, we can pass from S to T by adding the elements of T one by one and then removing elements of S one by one; at each step we have a generating family. Therefore we only have to show that if G has a minimal presentation over a finite generating family, then it has a

minimal presentation over a finite generating family obtained by adding one redundant generator, or by removing one redundant generator.

Suppose that $G = \langle S|\mathcal{R}\rangle$ with $\mathcal{R}$ minimal, and let t be an additional redundant generator. So $t = w(s_1, \ldots, s_k)$ with $s_i \in S$. Therefore a presentation over $S \cup \{t\}$ is given by

$$\langle S, t|\mathcal{R}, w(s_1, \ldots, s_k)t^{-1}\rangle.$$

This presentation is minimal. Indeed, clearly the last relator is not redundant, since once removed we obtain a presentation of the free product of G by the infinite cyclic group $\langle t\rangle$. Also, if $r \in \mathcal{R}$, then if the relator r is removed, we obtain

$$\langle S, t|\mathcal{R} \smallsetminus \{r\}, w(s_1, \ldots, s_k)t^{-1}\rangle$$

which also has the presentation

$$\langle S|\mathcal{R} \smallsetminus \{r\}\rangle;$$

this presentation cannot be a presentation of G by the assumed minimality of $\mathcal{R}$.

The other direction is slightly more tricky. Assume that $G = \langle S|\mathcal{R}\rangle$ with $\mathcal{R}$ minimal, and let us suppose that some $t \in S$ is redundant in G, say $t = w(s_1, \ldots, s_k)$ with $s_i \in S' = S \smallsetminus \{t\}$. There exists a finite subset $\mathcal{R}_0 \subset \mathcal{R}$ such that $t = w(s_1, \ldots, s_k)$ is a consequence of $\mathcal{R}_0$. Also, for $r \in \mathcal{R}$, let r' be the element obtained by substituting every occurrence of t by $w(s_1, \ldots, s_k)$, and $\mathcal{R}'$ the family $\{r' : r \in \mathcal{R}\}$. Adding an extra redundant relator, the group G has a presentation given by

$$\langle S|\mathcal{R}, w(s_1, \ldots, s_k)t^{-1}\rangle$$

and then by substitution we deduce the presentation

$$G = \langle S', t|\mathcal{R}', w(s_1, \ldots, s_k)t^{-1}\rangle;$$

since t occurs only in the last relator, we immediately see that this yields

$$G = \langle S'|\mathcal{R}'\rangle. \tag{4.1}$$

We claim that if $r \in \mathcal{R}$ and r' is a redundant relator in (4.1) then $r \in \mathcal{R}_0$. Indeed, if $r \notin \mathcal{R}_0$, we can start from the presentation $G_r = \langle S|\mathcal{R} \smallsetminus \{r\}\rangle$ and since $r \notin \mathcal{R}_0$, the word $w(s_1, \ldots, s_k)t^{-1}$ is a relation in G_r so we can argue exactly as in the previous lines to see that

$$G_r = \langle S'|\mathcal{R}' \smallsetminus \{r'\}\rangle.$$

So from the minimality of the presentation of G, we deduce that r' is not redundant in (4.1). Accordingly, in the presentation (4.1), the set of redundant relators is finite and Lemma 4.6 applies. Thus G has a minimal presentation over $S' = S \smallsetminus \{t\}$. □

The notion of infinitely minimally presented groups is at first sight more natural than the notion of INIP groups; however, it is much less convenient to deal with. For instance, let a finitely generated group Γ decompose as a direct product $\Gamma = \Gamma_1 \times \Gamma_2$. It is immediate that if Γ_1 is INIP, then so is Γ. Is it true that if Γ_1 is infinitely minimally presented, then so is Γ?

It turns out that there exist examples of INIP groups with no minimal presentation, but the only one we could see are based on a delicate construction by Ould Houcine, see Example 4.23. It would be interesting to find other examples. For instance, in §4.B, we prove that for any infinite finitely generated group G, the wreath product $\mathbf{Z} \wr G$ is INIP. The proof also shows that if G is minimally presented, then $\mathbf{Z} \wr G$ is infinitely minimally presented as well. However, for G arbitrary, we do not know if $\mathbf{Z} \wr G$ is always minimally presented.

4.B. **Various examples of continuously independent families.** There is a vast
source of independent families in the literature; many of them are actually continuously
independent. Each of these families provides an explicit embedding of the Cantor set in
the space of marked groups.

4.B.1. *Wreath products and wreathed Coxeter groups.* We first introduce the following
definition.

Definition 4.7. We say that a finitely generated group H is *largely related* if for every
epimorphism $G \twoheadrightarrow H$ of a finitely presented group G onto H, the kernel N admits a
non-abelian free quotient.

Clearly, a largely related group cannot be finitely presented; the converse is not true
because the quotient of any finitely presented group by an infinitely generated central
subgroup is neither infinitely presented nor largely related. Consequences of being largely
related are given in §5.C.1.

Definition 4.8. Let G be a group and $H \subset G$ a subgroup. Consider the equivalence
relation on G: $g_1 \sim g_2$ if g_1 belongs to the same H-double coset as g_2 or g_2^{-1}. Let Q be
the quotient of G by this equivalence relation and $Q^* = Q \smallsetminus \{H\}$ (observe that H is a
single equivalence class).

Let B be another group and define Γ as the quotient of $B * G$ by the normal closure
of $[B, H]$. Observe that in Γ, the normal subgroup N_g generated by $[gBg^{-1}, B]$, for
$g \in G \smallsetminus H$, only depends on the class of g in Q^*.

Proposition 4.9. *If $B \neq 1$ and $H \neq G$, then the family of normal subgroups (N_g) of
Γ is continuously independent, when g ranges over Q^*. In particular, if the double coset
space $H\backslash G/H$ is infinite and both B and G are finitely generated, then the permutational
wreath product $B \wr_{G/H} G = B^{(G/H)} \rtimes G$ is an infinitely independently presented group and
is largely related.*

This extends a result of [Cor06], where it was shown under the same assumptions that
the wreath product $B \wr_{G/H} G$ is infinitely presented.

Proof. Actually, we do not need to assume any finitely generation assumption, and the
general statement we prove is that $B \wr_{G/H} G$ is infinitely independently presented and
largely related relative to Γ as soon as $H\backslash G/H$ is infinite $B \neq 1$, $H \neq G$. (If $\Gamma \twoheadrightarrow \Lambda$ is
an arbitrary epimorphism between groups, we say here that Λ is largely related relative
to Γ if the kernel has a nonabelian free quotient.) If G, H are finitely generated then Γ
is finitely generated, so by picking a finitely presented subgroup with an epimorphism
$\Gamma_0 \twoheadrightarrow \Gamma$, we deduce that G is infinitely independently presented and largely related.
First observe that Γ has an obvious description as the semidirect product

$$B^{*(G/H)} \rtimes G,$$

where $U = B^{*(G/H)}$ is a free product of copies B_x of B indexed by G/H, with the obvious
shifting action of G. From this it already follows that the quotient of Γ by the subgroup
N_{Q^*} generated by all N_g (which is normal) is the permutational wreath product.
Observe that Q^* is obtained from $H\backslash G/H$ by removing one point and modding out
an action of the cyclic group of order two (by inversion). So if $H\backslash G/H$ is infinite, so is
Q^*. So once we shall have proved that (N_g) is independent, it follows that the quotient
$B \wr_{G/H} G$ is INIP.
Lift Q^* to a subset of G. If I is a subset of Q^*, let N_I be the normal subgroup of G
generated by $\bigcup_{g \in I} N_g$. Then N_I is generated, as a normal subgroup of G, by $\bigcup_{g \in I} [B_{gH}, B]$.

That (N_g) is independent means that for every $s \in Q^*$, the group $N_{Q^* \smallsetminus \{s\}}$ is not equal to N_{Q^*}. We will actually show that for every $s \in Q^*$, the group $N_{Q^*}/N_{Q^* \smallsetminus \{s\}}$ is non-trivial.

The Reidemeister-Schreier method (or a straightforward direct verification) shows that $N_{Q^* \smallsetminus \{s\}}$ is generated, as a normal subgroup of U, by $\bigcup_{g \in Q^* \smallsetminus \{s\}, \gamma \in G} \gamma [B_{g\xi}, B_\xi] \gamma^{-1}$ (where ξ denotes the base-point in G/H), or equivalently by

$$\bigcup_{g \in Q^* \smallsetminus \{s\}} \bigcup_{\gamma \in G} [B_{\gamma g\xi}, B_{\gamma\xi}]. \tag{4.2}$$

Let J_s be the normal subgroup of U generated by

$$\bigcup_{x \in G/H \backslash \{\xi, s\xi\}} B_x,$$

so U/J_s is naturally identified with $B_\xi * B_{s\xi}$. Observe that $N_{Q^* \smallsetminus \{s\}}$ is contained in J_s: indeed it is generated by commutators $[B_x, B_y]$ with $\{x, y\} \neq \{\xi, s\xi\}$, so each of these commutators is contained in J_s. Thus there is a natural epimorphism $U/N_{Q^* \smallsetminus \{s\}} \twoheadrightarrow B_\xi * B_{s\xi}$. It restricts to an epimorphism

$$N_{Q^*}/N_{Q^* \smallsetminus \{s\}} \twoheadrightarrow K,$$

where K is the kernel of the projection $B_\xi * B_{s\xi} \twoheadrightarrow B_\xi \times B_{s\xi}$.

Since $B \neq 1$, K is a nontrivial free group and in particular $N_{Q^*}/N_{Q^* \smallsetminus \{s\}}$ is non-trivial. If moreover B has at least 3 elements, this shows that $N_{Q^*}/N_{Q^* \smallsetminus \{s\}}$ surjects onto a non-abelian free group. If B has two elements, this can be proved to be still true under the mild assumption that G/H has at least 4 elements, but the above proof directly shows something easier, namely that if s, t are distinct in Q^*, the group $N_{Q^*}/N_{Q^* \smallsetminus \{s,t\}}$ surjects onto a non-abelian free group, namely the kernel of the projection $B_\xi * (B_{s\xi} \times B_{t\xi}) \twoheadrightarrow B_\xi \times B_{s\xi} \times B_{t\xi}$. This shows that, whenever $B \neq 1$, for every subset $I \subset Q^*$ whose complement contains at least two elements, N_{Q^*}/N_I has a non-abelian free quotient.

Thus if Q^* is infinite, and if P is any finitely presented group with an epimorphism π onto $B \wr_{G/H} G$, this epimorphism factors through the projection Γ/N_I for some finite $I \subset Q^*$. So, the kernel of π admits N_{Q^*}/N_I as a quotient and therefore possesses a non-abelian free group as a quotient. This shows that $B \wr_{G/H} G$ is largely related.

It remains to check that the family $(N_g)_{g \in Q^*}$ is continuously independent, using the criterion of Lemma 4.2. Let s be an element in $\Gamma \smallsetminus N_I$ and let us find a cofinite subset P containing I such that $s \notin N_P$. If $s \notin U$ we can pick $P = Q^*$, so assume $s \in U$. More precisely, s belongs to the subgroup generated by the U_x where x ranges over some finite subset F of G/H. Define $J' = \{g \in G | \exists x, x' \in F : gx = x'\}$. Let J be its image in Q; then J is finite: indeed, for each $(x, x') \in (G/H)^2$ the set of g such that $gx = x'$ is equal to a single double coset of H. Set $P = I \cup (Q^* \smallsetminus J)$. We claim that $s \notin N_P$. We use a retraction argument similar to the previous one. Define U^F as the quotient of the free product U by all B_x for $x \notin F$; this is naturally identified to the free product of B_x for $x \in F$; let U_I^F be the image of U^F in U/N_I. Let us check that the epimorphism $U/N_I \twoheadrightarrow U/N_P$ is injective in restriction to U_I^F. Indeed, U/N_P is obtained from U/N_I by modding out by the elements $[B_{\gamma g\xi}, B_{g\xi}]$ for $g \in G$ and $\gamma \in Q^* \smallsetminus J$, and by definition of J, those elements have trivial image in U^F, making the following diagram

$$\begin{array}{ccc} U/N_I & \longrightarrow & U/N_P \\ \uparrow & & \uparrow \\ U_I^F & = \!\!=\!\!= & U_I^F, \end{array}$$

commutative and we conclude that $s \notin N_P$. $\qquad\qquad\square$

Example 4.10. Let $F_2 = \langle t, x \rangle$ be the free group on 2 generators. By Proposition 4.9, the sequence $(u_n)_{n>0}$ with $u_n = [t^n x t^{-n}, x]$ is continuously independent, giving rise to an infinite minimal presentation of the wreath product $\mathbf{Z} \wr \mathbf{Z}$. That this presentation is independent was established (with a much longer proof) by G. Baumslag [Bau61], see also [Str84, Thm. 4].

We now turn to another similar example based on Coxeter groups. Let G be a group and V a transitive G-set, with base-point v_0 with stabilizer H. Consider a *Coxeter matrix* on V i.e. a symmetric matrix μ with diagonal entries equal to 1 and non-diagonal entries in $\{2, 3, \dots, \infty\}$. It defines the Coxeter group with Coxeter presentation

$$W(V, \mu) = \left\langle (w_v)_{v \in V} \mid \left((w_s w_t)^{\mu(s,t)} \right)_{(s,t) \in V^2} \right\rangle.$$

Now assume that μ is G-invariant, in the sense that $\mu(gs, gt) = \mu(s, t)$ for all $g \in G$ and $(s, t) \in V^2$. This induces a natural action of G by automorphisms on $W(V, \mu)$, so that $g \cdot w_s = w_{gs}$. The corresponding semidirect product

$$W(V, \mu) \rtimes G$$

is called a *wreathed Coxeter group*. This group was already considered, from a different perspective, in [CSV12]. By transitivity of the G-action on V, it is generated by G and a single w_s, and in particular is finitely generated whenever G is finitely generated.

We consider the graph X with vertex set V and a non-oriented edge between x and y whenever $1 < \mu(x, y) < \infty$, labeled by $\mu(x, y)$. By the G-invariance of μ, the action of G acts on V preserves the graph structure and the labeling. Set $G^1 = \{g \in G | 1 < \mu(v_0, gv_0) < \infty\}$; it is a subset, stable under the equivalence relation $\sim$ from Definition 4.8; let $Q^1 = G^1/\sim$ be the quotient set; it can be identified to the set of non-oriented edges of X modulo the action of G.

Start from the group $\Gamma = (G * \mathbf{Z}/2\mathbf{Z})/[H, w]$, where $\mathbf{Z}/2\mathbf{Z} = \langle w \mid w^2 = 1 \rangle$; it has a natural decomposition $(\mathbf{Z}/2\mathbf{Z})^{*G/H} \rtimes G$. If $g \in G^1$, denoting ${}^g w = gwg^{-1}$, we define N_g as the normal subgroup of Γ generated by $(w\, {}^g w)^{\mu(v_0, gv_0)}$; clearly N_g only depends on the equivalence class of $g \in Q^1$ and for $I \subset Q^1$, define N_I as the subgroup generated by $\bigcup_{g \in I} N_g$, so that

$$\begin{aligned}
\Gamma/N_{Q^1} &\to W(V, \mu) \rtimes G \\
G \ni g &\mapsto g \\
w &\mapsto w_{v_0}
\end{aligned}$$

is an isomorphism.

Proposition 4.11. *The family $(N_g)_{g \in Q^1}$ of normal subgroups of Γ is continuously independent. In particular, if G is finitely generated and Q^1 is infinite (i.e. if the set of G-orbits of edges in X is infinite), then the wreathed Coxeter group $W(V, \mu) \rtimes G$ is infinitely independently presented and largely related.*

The proof of Proposition 4.11 is similar in many respects to that of Proposition 4.9. However, we need the following well-known theorem, due to Tits, about Coxeter groups [Hum90, Thm. 5.5].

Theorem 4.12 (Tits)**.** *Given a Coxeter group generated by involutions $(\sigma_s)_{s \in S}$, subject to relators $(\sigma_s \sigma_t)^{\mu(s,t)}$ for all s, t (where $\mu(s, t)_{(s,t) \in S \times S}$ is a Coxeter matrix), the element $\sigma_s \sigma_t$ has order exactly $\mu(s, t)$, and every subgroup generated by a subset $(\sigma_s)_{s \in T}$ is a Coxeter group over this system of generators.*

When the Coxeter matrix has all its non-diagonal entries even (possibly ∞), this can be proved by a trivial retraction argument as the one we use in the truncated presentations of wreath products (see the proof of Proposition 4.9). However this argument falls apart if the Coxeter matrix has some odd non-diagonal entry.

Proof of Proposition 4.11. As in the proof of Proposition 4.9, we do not need to assume that G is finitely generated, and the general statement is that the wreathed Coxeter group is infinitely independently presented and largely related relative to Γ as soon as Q^1 is infinite (see the first lines of the proof of Proposition 4.9). If G is finitely generated then Γ is finitely generated and the result follows.

Because of the similarity with the proof of Proposition 4.9, we will prove the result in a particular case that is enough to encompass all the differences, namely the case when $G = \mathbf{Z} = \langle t \rangle$ and $H = \{0\}$. The reader is invited to prove the general case as an exercise.

So we have to prove that in the free product $\Gamma = \langle t, w | w^2 = 1 \rangle$, the family of relators $r_n = (wt^n wt^{-n})^{\mu(0,n)}$, for $n \in \mathbf{Z}^1 = \{n \geq 1 : \mu(0,n) < \infty\}$, is continuously independent.

If $p \in \mathbf{Z}^1$, let $\Gamma_{[p]}$ be the group obtained by modding out Γ by all relators r_n for $n \neq p$, and let μ' be the matrix obtained from μ by replacing all entries $\mu(n, n+p)$ by ∞. We see that in $\Gamma_{[p]} = W(V, \mu') \rtimes \mathbf{Z}$. By Tits' theorem, $wt^p wt^{-p}$ has infinite order in $\Gamma_{[p]}$, so $r_p \neq 1$ in $\Gamma_{[p]}$. This proves independency of the family of relators.

Now let us check continuous independency at every $I \subset \mathbf{Z}^1$. Define $w_i = t^i wt^{-i}$, so that N_I is by definition the normal subgroup of Γ generated by elements of the form $(w_i w_j)^{\mu(i,j)}$, where (i, j) ranges over pairs in $\mathbf{Z}^2$ such that $i - j \in I$.

Let s be an element in $\Gamma \smallsetminus N_I$ and let us find a cofinite subset P of $\mathbf{Z}^1$, containing I, such that $s \notin N_P$. If $s \notin U$, the normal subgroup of Γ generated by w, then $P = \mathbf{Z}^1$ works, so assume $s \in U$. Note that U is a free product of copies $\langle w_i \rangle$ of the cyclic group of order two, indexed by $i \in \mathbf{Z}$. Let F be a finite subset of $\mathbf{Z}$ such that s belongs to the subgroup generated by w_i for $i \in F$. Let J be the set of positive elements of the form $i - j$ for $i, j \in F$, and set $P = \mathbf{Z}^1 \smallsetminus (J \smallsetminus I)$, which is cofinite in $\mathbf{Z}^1$. Then, by Tits' theorem, in U/N_P (resp. in U/N_I), the subgroup generated by the w_i for $i \in F$ is a Coxeter group over this family of generators, with Coxeter exponents, for $i \neq j$, given by $m(i,j) = \mu(i,j)$ if $|i - j| \in P$ (resp., in U/N_I, if $|i - j| \in I$) and $m(i,j) = \infty$ otherwise. But for $i, j \in F$, the condition $|i - j| \in P$ and $|i - j| \in I$ are equivalent, by definition of P. Thus the family $(w_i)_{i \in F}$ is subject to the same relations in U/N_I and in U/N_P, and therefore $s \notin N_P$. So Lemma 4.2 applies, and the map Φ associated to $(N_n)_{n \in \mathbf{Z}^1}$ is continuous at I. $\square$

Example 4.13. The group of translations of $\mathbf{Z}$ generated by the transposition $0 \leftrightarrow 1$ and the shift $n \mapsto n + 1$, which is isomorphic to wreathed Coxeter group $\mathrm{Sym}_0(\mathbf{Z}) \rtimes \mathbf{Z}$, is infinitely independently presented. That this group is infinitely presented is well known; it was mentioned by Stëpin [Stë83] as an example of a finitely generated group that is approximable by finite groups (for the topology of $\mathcal{G}_m$) but is not residually finite, while it is observed in the same paper (see also [VG97]) that these conditions cannot be fulfilled by a finitely presented group.

4.B.2. *Small cancellation groups.* We now indicate a very rich source of examples, namely groups satisfying the $C'(1/6)$ small cancellation condition. Start from a free group F_S (for the moment the free generating set S is allowed to be infinite). Given two words $w, x \in F_S$, we say that w is a *piece* of x if some cyclic conjugate of w is a (convex) subword in some cyclic conjugate of x or x^{-1}. If $|w| \geq |x|/6$ it is called a $1/6$-*large piece*, and if $|w| > |x|/2$ it is called an *essential piece*. Let (u_n) be a (finite or infinite) sequence of nontrivial elements in F_S. It is said to satisfying the $C'(1/6)$-condition, if any word w occurring as a $1/6$-large piece in one of the words u_n does not appear as a piece anywhere

else, (i.e. does not reappear as another piece in u_n, or as a piece in any u_m for $m \neq n$). Note that, by definition, the condition implies that if u_n has length k, then the $2k$ words occurring as cyclic conjugates of u_n and u_n^{-1} are pairwise distinct, or equivalently that no u_n is a proper power. When $\#S \geq 2$, examples of infinite sequences (u_n) satisfying the $C'(1/6)$-condition are given in [LS77, p. 283].

The $C'(1/6)$-condition was cooked up so that the following lemma holds [LS77, Thm. V.4.4].

Lemma 4.14 (Greendlinger). *If (u_n) is a (finite or not) family satisfying the $C'(1/6)$-condition, then every x in the normal subgroup N generated by the u_i's has a cyclic conjugate containing an essential piece, i.e. a piece of some u_i of size $> |u_i|/2$.*

Remark 4.15. One essential motivation for Lemma 4.14 was to give a fast algorithmic solution to the word problem (when S and the family (u_n) are finite), by the so-called *Dehn algorithm.* Namely, given x, we wish to determine whether $x \in N$. We thus check if x or some cyclic conjugate contains an essential piece of one of the relators. If no, the process stops. If yes, x contains an essential piece w of some relator u_i; replace in x the piece w by the inverse of its complement in u_i, yielding a word of size $\leq |x| - 1$, and apply the algorithm once again. The process stops after at most $|x|$ steps; if the resulting word is the trivial word, then $x \in N$ and otherwise $x \notin N$ by the Greendlinger lemma. Originally, Dehn proved this algorithm is valid for surface groups of genus $g \geq 2$, with their standard one-relator presentation

$$\Gamma_g = \left\langle x_1, \ldots, x_g, y_1, \ldots, y_g \,\middle|\, \prod_{i=1}^{g} [x_i, y_i] \right\rangle.$$

Lemma 4.16. *Let F_S be a free group and $(u_i)_{i \in I}$ a sequence satisfying the $C'(1/6)$-condition. Then the family (u_n) is continuously independent.*

The independency was explicitly observed in [Gre61].

Proof. It is obvious that the standard map Φ (see Definition 4.1) is order-preserving and lower semicontinuous. By the $C'(1/6)$-condition and Lemma 4.14, u_i does not belong to the normal subgroup generated by $\{u_j | j \neq i\}$ (i.e. $U_{I \smallsetminus \{i\}}$) and it immediately follows that Φ is injective. The continuity is due to the same reason, albeit slightly refined. Let $J \subset I$ be a subset and let us check upper semicontinuity at J, using the criterion in Lemma 4.2. Let w be an element of $F_S \smallsetminus U_J$ of minimal length and let us check that there exists a cofinite subset P containing J such that $w \notin U_P$. Namely, define P as the union of J and of the set J' of all j such that u_j does not contain an essential piece which is a piece of w. By the $C'(1/6)$-condition, a given piece of w is contained as an essential piece in at most one u_n, and since w contains finitely many pieces, we deduce that J' (and hence P) is cofinite in I. We wish to check that $w \notin U_P$. Indeed, otherwise, by Lemma 4.14, there exists $n \in P$ such that some piece of w is an essential piece of u_n. By definition of J', we have $n \notin J'$, and therefore $n \in J$. Hence replacing in w this piece by the inverse of its complement in u_n, the resulting element w' also belongs to $F_S \smallsetminus U_J$ and has smaller size, a contradiction. So $w \notin U_P$. This proves upper semicontinuity. $\square$

Remark 4.17. Lemma 4.16 might generalize, by small cancellation theory over hyperbolic groups (e.g. the methods of [Ols93]), to the statement that for every nonelementary word hyperbolic group contains an infinite, continuously independent family (u_n). (Compare with §5.B.1.)

4.B.3. *Schur multiplier.* Let us now mention a connection between the multiplier $H_2(G, \mathbf{Z})$ of a group, and infinite independent presentability. We first use the following probably

well-known lemma. Recall that an abelian group is called *minimax* if it has a finitely generated subgroup B such that A/B is artinian, i.e. satisfies the descending chain condition on subgroups.

Lemma 4.18. *Let A be an abelian group. The following are equivalent*

    (1) A is not minimax;
    (2) the poset $(\mathcal{N}(A), \subset)$ contains a subposet isomorphic to $(\mathbf{Q}, \leq)$;
    (3) the poset $(\mathcal{N}(A), \subset)$ contains a subposet isomorphic to the power set $(2^{\mathbf{Z}}, \subset)$;
    (4) A has an infinite independent sequence of (normal) subgroups;
    (5) A has a quotient that is an infinite direct sum of nonzero groups.

Proof. That (5) implies (4) is trivial. It directly follows from Definition 4.1 that (4) implies (3). Also, identifying $\mathbf{Z}$ to $\mathbf{Q}$ and considering a chain of intervals proves that (3) implies (2).

Suppose (1) and let us prove (5). We first claim that A has a quotient that is torsion and not minimax.

Indeed, let $(e_i)_{i \in I}$ be a maximal $\mathbf{Z}$-free family in A, generating a free abelian group B. Then $A/2B$ is torsion and is not minimax: indeed if I is finite, $2B$ is minimax but not A, and since minimax is stable under taking extensions, $A/2B$ cannot be minimax; if I is infinite, $A/2B$ admits $B/2B \simeq (\mathbf{Z}/2\mathbf{Z})^{(I)}$ as a subgroup so is not minimax.

Thus we can, replacing A by this quotient, suppose that A is torsion. If A admits p-torsion for infinitely many p's, then it is itself an infinite direct sum (of its p-components) and we are done. Otherwise, A has p-torsion for finitely many p's only, so taking another quotient, we can suppose that A is a p-group for some prime p.

If A/pA is infinite, it an infinite direct sum of copies of $\mathbf{Z}/p\mathbf{Z}$ and we are done. Otherwise, let F be the subgroup generated by a transversal of pA in A; F is finite and A/F is divisible and non-minimax, hence isomorphic to $\mathbf{Z}(p^\infty)^{(I)}$ for some infinite I.

(Note that the proof readily shows that an arbitrary non-minimax abelian group A has a quotient which is isomorphic to either $(\mathbf{Z}/p\mathbf{Z})^{(I)}$, $\mathbf{Z}(p^\infty)^{(I)}$ for some infinite set I and some prime p, or a product $\prod_{p \in P} \mathbf{Z}/p\mathbf{Z}$ or $\prod_{p \in P} \mathbf{Z}(p^\infty)$ for some infinite set P of primes.)

To conclude, we need to prove (2)$\Rightarrow$(1). Assuming (2), there is an increasing chain $(L_n)_{n \in \mathbf{Q}}$ of subgroups of A. Since for all n there is an infinite chain of subgroups between L_{n+1} and its subgroup L_n, we deduce that L_n has infinite index in L_{n+1}. To show that A is not minimax, let us check that in contrast, any minimax abelian group B satisfies the property that for every nondecreasing sequence (B_n) of subgroups, there exists n_0 such that B_n has finite index in B_{n+1} for all $n \geq n_0$. Indeed, this property is stable under taking group extensions, and is obviously satisfied by $\mathbf{Z}$ and $\mathbf{Z}(p^\infty)$, and finite groups, so is satisfied by all minimax groups. Thus A is not minimax. $\square$

Note that a non-minimax abelian group can be very far from being or containing a nontrivial infinite direct sum, e.g. $\mathbf{Q}$ is such an example.

Proposition 4.19. *Let G be a finitely generated group and assume that the Schur multiplier $H_2(G, \mathbf{Z})$ is not minimax. Then G is infinitely independently presented.*

Proof. Let F be a finitely generated free group with an epimorphism $\pi : F \twoheadrightarrow G$, let R denote its kernel. The extension $R \rightarrowtail F \twoheadrightarrow G$ gives rise to an exact sequence in homology

$$H_2(F, \mathbf{Z}) \to H_2(G, \mathbf{Z}) \to R/[R, F] \xrightarrow{\pi_{\mathrm{ab}}} F_{\mathrm{ab}} \twoheadrightarrow G_{\mathrm{ab}}$$

(see for instance [HS97, Cor. 8.2]). Since F is free, the multiplier $H_2(F, \mathbf{Z})$ is trivial, so the above exact sequence leads to the extension

$$H_2(G, \mathbf{Z}) \rightarrowtail R/[R, F] \twoheadrightarrow \mathrm{Ker}(\pi_{\mathrm{ab}}).$$

Thus $R/[R, F]$ is not minimax, so by Lemma 4.18 it admits an infinite direct sum (A_n) of nontrivial groups as a quotient. If (M_n) is the inverse image of A_n in F, then it is an infinite sequence of independent normal subgroups of F generating R, so $F/R = G$ is INIP. $\square$

Example 4.20. Examples of finitely generated groups satisfying the assumptions of Proposition 4.19 (and thus INIP) are groups of the form $G = F/[R, R]$, where F is free and R is an arbitrary normal subgroup of infinite index. Indeed, it was shown in [BST80] that $H_2(G, \mathbf{Z})$ then has a free abelian quotient of infinite rank.

Example 4.21. The first Grigorchuk group Γ is an example of a group with growth strictly between polynomial and exponential [Gri85]. In [Gri99], Grigorchuk proves that Γ has an infinite minimal presentation, and more precisely that the "Lysënok presentation" is minimal, by showing that it projects to a basis of the Schur multiplier $H_2(G, \mathbf{Z})$, which is an elementary 2-group of infinite rank.

We now provide examples of groups that are INIP but not minimally presented.

Proposition 4.22. *Let G be a finitely presented group and Z a central subgroup. Assume that Z is divisible and non-trivial. Then G/Z is not minimally presented.*

Proof. Since Z is divisible and non-trivial, it is not finitely generated and therefore G/Z is infinitely presented. Let F be a free group of finite rank with an epimorphism $F \twoheadrightarrow G$, with kernel K. Assume by contradiction that there exists a minimal family $(u_n)_{n \geq 0}$ of relators for G/Z over F. Since G/Z is infinitely presented, the family (u_n) is infinite. Since K is finitely generated as a normal subgroup, it is contained in the normal subgroup generated by $u_0, \dots, u_k$ for some k. Let M be the normal subgroup of F generated by all u_i for $i \neq k+1$. Then M contains K, and thus M/K can be identified with a subgroup Z' of Z, so that Z/Z' is generated by the image of u_{k+1}. But as a quotient of Z, it is divisible and cyclic, hence trivial. So $Z' = Z$, i.e. M is the kernel of $F \twoheadrightarrow G/Z$. This contradicts the minimality of the family of relators (r_n). $\square$

Example 4.23. Ould Houcine [OH07] showed that every countable abelian group embeds into the centre of a finitely presented group. So there exist examples where Proposition 4.22 applies, with $Z = \mathbf{Q}$ (or any other countable divisible non-minimax abelian group), yielding that G/Z is not minimally presented. On the other hand, by implication $(1) \Rightarrow (4)$ of Lemma 4.18 the group G/Z is INIP.

Another application of Ould Houcine's result is when $Z \simeq \mathbf{Q}/\mathbf{Z}$. In that case, since Z is torsion and divisible, we have $Z \otimes_{\mathbf{Z}} A = \{0\}$ for every abelian group A that is either torsion or divisible, e.g. when A is the underlying abelian group of an arbitrary field. It follows that G/Z is of type FP_2 over every field. (Compare with Proposition 5.9.)

4.B.4. *More examples.*

Example 4.24. Let us mention another family of groups obviously having a continuously independent sequence of normal subgroups (and, anticipating on Section 5, are thus of intrinsic condensation by Lemma 5.2): those groups G having a normal subgroup decomposing as an infinite direct sum $\bigoplus H_n$, each H_n being a nontrivial normal subgroup of G. Clearly the family (H_n) (or any family (h_n), h_n being a nontrivial element of H_n) is then continuously independent. Examples include

- Any group G whose centre Z contains an infinite direct sum. Finitely presented examples (which are in addition solvable) appear in [BGS86, Sec. 2.4].
- Finitely generated groups introduced by B.H. Neumann, containing an infinite direct sum of finite alternating groups $\bigoplus H_n$ as normal subgroup [Neu37, Thm. 14] (with each H_n normal in G).

- Any groups isomorphic to proper direct factor of itself, i.e., G isomorphic to $G \times H$ where H is a nontrivial group: indeed, if ϕ is an injective endomorphism of $H \times G$ with image $\{1\} \times G$, then we can set $H_n = \phi^n(H \times \{1\})$, and clearly H_n is normal and the H_n generate their direct sum. Finitely generated examples of such groups (actually satisfying the stronger requirement $G \simeq G \times G$) are constructed in [Jon74]. Note that a group isomorphic to a proper direct factor is in particular isomorphic to a proper quotient, i.e. is *non-Hopfian*. However, a finitely generated non-Hopfian group does not necessarily contain a normal subgroup that is an infinite direct sum of non-trivial normal subgroups (anticipating again on Section 5, it is not even necessarily of intrinsic condensation). Indeed, if $n \geq 3$ and A_n is the Abels' group introduced in Section 3 and Z_0 an infinite cyclic subgroup of its centre, then A_n/Z_0 is not Hopfian (the argument is due to P. Hall for A_3 and extends to all n [Hal61, Abe79]), but Lyulko's result (Proposition 3.4) implies that A_n has only countably many subgroups.

4.C. **Independent varieties.** Let us now give a much less trivial, and powerful example, independently due to Adian, Olshanskii and Vaughan-Lee [Adj70, Ols70, VL70]. It will be used in a crucial way in the proof of Theorem 1.7.

Theorem 4.25 (Adian, Olshanskii, Vaughan-Lee). *If F is a free group over at least two generators (possibly of infinite rank), then F has uncountably many fully characteristic subgroups (i.e. stable under all endomorphisms). Actually, it has an independent infinite sequence of fully characteristic subgroups.*

Remark 4.26. It is conceivable that Theorem 4.25 can be improved to the statement that F has a *continuously* independent sequence of fully characteristic subgroups.

Corollary 4.27. *If H is any group admitting an epimorphism onto a non-abelian free group F, then H has at least continuously many fully characteristic subgroups.*

Proof. Our aim is to construct, given a fully characteristic subgroup N of F, a fully characteristic subgroup $\Phi(N)$ of H in such a way that the function $N \mapsto \Phi(N)$ is injective. Theorem 4.25 will then imply the conclusion of Corollary 4.27.

We begin by fixing the notation. For every group G, let $\mathcal{FCH}(G)$ denote the set of its fully characteristic groups. Next, given a group L, let $U_G(L)$ denote the intersection of the kernels of all homomorphisms $g \colon G \to L$ from G into L; in symbols

$$U_G(L) = \bigcap \{\ker g \mid g \colon G \to L\}.$$

It is easily checked that $U_G(L)$ is a fully characteristic subgroup of G.

We apply this construction twice: first with $G = H$ and $L = F/N$ where $N \in \mathcal{FCH}(F)$ and obtain the group $\Phi(N) \in \mathcal{FCH}(H)$; in symbols,

$$\Phi(N) = U_H(F/N) = \bigcap \{\ker h \mid h \colon H \to F/N\}. \tag{4.3}$$

Secondly, we apply it with $G = F$ and $L = H/M$ where $M \in \mathcal{FCH}(H)$, obtaining the fully characteristic subgroup

$$\Psi(M) = U_F(H/M) = \bigcap \{\ker f \mid f \colon F \to H/M\} \tag{4.4}$$

of F. We claim that $\Psi \circ \Phi$ is the identity of $\mathcal{FCH}(F)$.

Given $N \in \mathcal{FCH}(F)$, set $M = \Phi(N)$ and $N' = \Psi(M)$. We first show that $N' \subseteq N$. Since F is free, the projection $\pi \colon H \twoheadrightarrow F$ admits a section $\iota \colon F \rightarrowtail H$. By (4.4), the group $N' = \Psi(M)$ is contained in the kernel of the homomorphism $f_0 = \mathrm{can}_M \circ \iota \colon F \rightarrowtail$

$H \twoheadrightarrow H/M$. Next, by (4.3), the group M is contained in the kernel of the map $h_0 = \mathrm{can}_N \circ \pi \colon H \twoheadrightarrow F \twoheadrightarrow F/N$. Since $\ker h_0 = \pi^{-1}(N)$ it follows that

$$N' \subseteq \ker f_0 \subseteq \ker(F \rightarrowtail H \twoheadrightarrow H/(\pi^{-1}(N)) = \iota^{-1}\left(\pi^{-1}(N)\right) = (\pi \circ \iota)^{-1}(N) = N.$$

The opposite inclusion $N \subseteq N'$ will be proved by showing that $F \smallsetminus N' \subseteq F \smallsetminus N$. Suppose that $x \in F \smallsetminus N' = F \smallsetminus \Psi(M)$. Then there exists a homomorphism $f \colon F \to H/M$ such that $f(x) \neq 1$. Since F is free, f can be lifted to a homomorphism $\tilde{f} \colon F \to H$ with $\tilde{f}(x) \notin M$. By the definition of $M = \Phi(N)$, there exists a homomorphism $h \colon H \to F/N$ such that $h(\tilde{f}(x)) \neq 1 \in F/N$. The composition $\varphi = h \circ \tilde{f} \colon F \to F/N$ then lifts to an endomorphism $\tilde{\varphi}$ of F with $\tilde{\varphi}(x) \notin N$. But N is fully characteristic, so it is mapped into itself by $\tilde{\varphi}$ and thus $x \in F \smallsetminus N$. $\qquad\square$

5. Condensation groups

5.A. **Generalities on condensation groups.** Recall that if G is a group, we denote by $\mathcal{N}(G)$ the set of its normal subgroups; this is a closed (hence compact) subset of 2^G.

We say that a finitely generated group G is

- of *condensation* if given some (and hence any) marking of G by m generators, G lies in the condensation part of $\mathcal{G}_m$;
- of *extrinsic condensation* if for every finitely presented group H and any epimorphism $H \twoheadrightarrow G$ with kernel N, there exist uncountably many normal subgroups of H contained in N;
- of *intrinsic condensation* if $\{1\}$ lies in the condensation part of $\mathcal{N}(G)$ (i.e. G lies in the condensation part of the set of its own quotients, suitably identified with $\mathcal{N}(G)$). The notion of intrinsic condensation was introduced in [CGP07, Sec. 6].

Note that being of intrinsic condensation makes sense even if G is not finitely generated. There is the following elementary lemma.

Lemma 5.1. *For every finitely generated group G, the following statements hold:*

  (i) *If G is of extrinsic condensation, then it is condensation and infinitely presented;*
  (ii) *if G is of intrinsic condensation, then it is condensation; if moreover G is assumed to be finitely presented, the converse is true.*

Proof. (i) It is clear from the definition that extrinsic condensation implies infinitely presented. To check that it also implies condensation, sssume $G \in \mathcal{G}_m$ is of extrinsic condensation and N is the kernel of the canonical epimorphism $F_m \twoheadrightarrow G$. Consider an open neighbourhood $\mathcal{O}_{\mathcal{F},\mathcal{F}'}$ of N in $\mathcal{G}_m$, with $\mathcal{F}$ and $\mathcal{F}'$ finite subsets of F_m as in (1.2). Define $N_{\mathcal{F}} \lhd F_m$ to be the normal subgroup generated by $\mathcal{F}$. Then $H = F_m/N_{\mathcal{F}}$ is a finitely presented group mapping onto G; since G is of extrinsic condensation, the kernel $N/N_{\mathcal{F}}$ of $H \twoheadrightarrow G$, and hence the neighbourhood $\mathcal{O}_{\mathcal{F},\mathcal{F}'}$, contain uncountably many normal subgroups. It follows that G is a condensation group.

(ii) Since given a marking of G there is a homeomorphic embedding of $\mathcal{N}(G)$ into $\mathcal{G}_m$ mapping $\{1\}$ to G, the first implication is obvious. The converse follows from the fact that if $G \in \mathcal{G}_m$ is finitely presented, then some neighbourhood of G consists of quotients of G. $\qquad\square$

Condensation and independency are related by the following fact.

Lemma 5.2. *Let G be group and $(M_i)_{i \in I}$ an independent family of normal subgroups. Then*

  (i) *if G is finitely generated and I is infinite then G/M_I is of extrinsic condensation. Thus INIP implies extrinsic condensation.*

(ii) if I is infinite and the mapping $\Phi : 2^I \to \mathcal{N}(G)$ associated to (M_i) is continuous at $\emptyset$, then G is of intrinsic condensation. In particular, if (M_i) is continuously independent, then for every co-infinite subset $J \subset I$, G/M_J is of intrinsic condensation.

Proof. We begin by (ii). We have $\Phi(\emptyset) = \{1\} \in \mathcal{N}(G)$. Since Φ is injective and continuous and 2^I is a perfect space, $\Phi(2^I)$ is a perfect space, so $\{1\}$ is a condensation point in $\mathcal{N}(G)$, i.e., G is of intrinsic condensation.

Let us now prove (i). Let H be a finitely presented group with an epimorphism $\pi : H \twoheadrightarrow G$. Let H_1 be a finitely presented group with an epimorphism $\rho : H_1 \twoheadrightarrow G$. Then the $\rho^{-1}(M_i)$ are also independent, so G/M_I is INIP relative to H_1. By Lemma 4.3, it follows G/M_I is also INIP relative to H, say by an infinite independent family of normal subgroups $(N_j)_{j \in J}$. The subgroups N_K, when K ranges over subsets of J, are pairwise distinct and contained in the kernel of π, and there are uncountably many such subgroups. By definition, this shows that G/M_I is of extrinsic condensation. $\square$

Remark 5.3. We therefore have

$$\text{Infinitely minimally presented} \Rightarrow \text{INIP} \Rightarrow$$

$$\text{Extrinsic Condensation} \Rightarrow \text{Infinitely Presented.}$$

The left implication is strict by Example 4.23. Examples 5.12 and 5.13, based on Abels' group, show that the two other implications are strict.

A useful corollary of Theorem 4.25 is the following.

Corollary 5.4. *Every group G with a non-abelian free normal subgroup has an infinite independent sequence of normal subgroups and is of intrinsic condensation.*

Proof. Let F be a non-abelian free normal subgroup in G. By Theorem 4.25, there is in F an infinite independent sequence (H_n) of characteristic subgroups; they are therefore normal in G and independent. In particular, F contains uncountably many normal subgroups of G. By Levi's theorem (see, e. g., [LS77, Chap. I, Prop. 3.3]) the derived subgroups $F^{(n)}$ (which are also free and non-abelian) intersect in the unit element, and therefore in $\mathcal{N}(G)$, every neighbourhood of $\{1\}$ contains the set of subgroups of $F^{(n)}$ for some n and is therefore uncountable.

If the free group F is countable, the preceding argument shows that $\{1\}$ lies in the condensation part of G. The argument fails if F is uncountable, since for a closed subset of 2^F, to show that an element is condensation it is not enough to show that all its neighbourhoods are uncountable. Let F be uncountable and let F' be a free factor of infinite countable rank, and $\mathcal{FCH}(F)$ denotes the set of fully characteristic subgroups of F, then the natural continuous map $c : \mathcal{FCH}(F) \to \mathcal{FCH}(F')$, defined by $c(N) = N \cap F'$, is injective. Indeed, let m be an element of F. Then there exists an automorphism of F (induced by a permutation of generators) mapping m to an element $m' \in F'$. This shows that for every characteristic subgroup N of F, we have $N = \bigcup_\alpha \alpha(N \cap F')$, where α ranges over automorphisms of F, and thus N is entirely determined by $N \cap F' = c(N)$. Thus by compactness, c is a homeomorphism onto its image. Since $\{1\}$ is a condensation point in $\mathcal{FCH}(F')$, we deduce that $\{1\}$ is a condensation point in $\mathcal{FCH}(F)$. $\square$

5.B. **Examples of intrinsic condensation groups.**

5.B.1. *Free groups and generalizations.* The free group F_2 is an example of an intrinsic condensation group; this is a well-known fact and follows, for instance, from Example 4.10 along with Lemma 5.2(ii).

More generally, having non-abelian free normal subgroups implies intrinsic condensation by Corollary 5.4. Note that this is much less easy, since both Corollary 5.4 and the

existence of free normal subgroups are then nontrivial facts. Here are some groups having nonabelian normal subgroups, and thus of intrinsic condensation.

- Non-abelian limits groups. These are finitely generated groups that are limits of non-abelian free groups in the space of marked groups. The existence of a non-abelian free subgroup follows from [CG05, Thm. 4.6].
- Non-elementary hyperbolic groups. The existence of a non-abelian normal free subgroup was established by Delzant [Del96, Thm. I]; the intrinsic condensation can also be derived from [Ols93, Thm. 3].
- Baumslag-Solitar groups

$$\mathrm{BS}(m,n) = \langle t, x \mid tx^m t^{-1} = x^n \rangle \qquad (|m|, |n| \geq 2);$$

indeed, the kernel of its natural homomorphism onto $\mathbf{Z}[1/mn] \rtimes_{m/n} \mathbf{Z}$ is free and non-abelian: it is indeed free because its action on the Bass-Serre tree of $\mathrm{BS}(m,n)$ is free. It is non-abelian, because otherwise $\mathrm{BS}(m,n)$ would be solvable, but it is a non-ascending HNN-extension so contains non-abelian free subgroups.
- Many other HNN-extensions obtained by truncating some group presentations, see Theorem 6.1.

Problem 5.5. Does there exist a non-ascending HNN-extension having no nontrivial normal free subgroup?

A natural candidate to be a counterexample would be a suitable HNN-extension of a non-abelian free group over a suitable isomorphism between two maximal subgroups. Recall that in [Cam53], simple groups are obtained as amalgams of non-abelian free groups over maximal subgroups of infinite index.

Another source of intrinsic condensation groups is Lemma 5.2(ii). It can be applied to the various examples of infinite continuously independent families of normal subgroups given in §4.B. For instance, from Example 4.10 we obtain that for every co-infinite set I of positive integers, the quotient of the free group $\langle t, x \rangle$ by the family of relators $[x, t^n x t^{-n}]$ for $n \in I$, is of intrinsic condensation.

5.B.2. *Intrinsic condensation and abelian groups.* The following is a particular case of [CGP10, Thm. G]. For the definition of minimax groups, see §4.B.3.

Proposition 5.6. *Let A be an abelian group. Then*
  (1) *A is a not a condensation point in $\mathcal{N}(A)$ if and only if A is minimax and does not admit $\mathbf{Z}(p^\infty)^2$ as a quotient for any prime p;*
  (2) *A is not of intrinsic condensation (i.e., $\{0\}$ is a not a condensation point in $\mathcal{N}(A)$) if and only if A is minimax and A/T_A does not admit $\mathbf{Z}(p^\infty)^2$ as a quotient for any prime p, where T_A is the torsion group of A.*

Note that the first condition also characterizes abelian groups with countably many subgroups, a result due to Boyer [Boy56] from which the first equivalence above can be checked directly.

Corollary 5.7. *If G is a group whose centre $Z(G)$ does not satisfy the condition in Proposition 5.6(2), then G is an intrinsic condensation group.* $\qquad\square$

Example 5.8. For all $n \geq 3$ and $k \geq 2$, if A_n is Abels' group then the direct power A_n^k is of intrinsic condensation. In contrast, A_n itself is not of intrinsic condensation by Proposition 3.4. Thus the class of groups that are not of intrinsic condensation is not closed under direct products, nor even direct powers.

5.C. **Examples of extrinsic condensation groups.**

5.C.1. *Largely related groups.* A rich source of examples of extrinsic condensation groups is given by the main result of this paper, namely that infinitely presented metabelian groups are extrinsic condensation groups. This will be established in Section 6, using the notion of largely related groups, introduced in Definition 4.7.

Proposition 5.9. *If a finitely generated group H is largely related, then it is of extrinsic condensation and is not (FP_2) over any nonzero commutative ring.*

Proof. Let us first prove the second assertion. We have to prove that for every nonzero commutative ring R and every epimorphism $G \twoheadrightarrow H$ with G finitely presented and with kernel N, we have $N_{\mathrm{ab}} \otimes_{\mathbf{Z}} R \neq 0$. Indeed, since N has a non-abelian free quotient, N_{ab} admits $\mathbf{Z}$ as a quotient and so $N_{\mathrm{ab}} \otimes_{\mathbf{Z}} R \neq 0$.

To check that H is of extrinsic condensation, let $p : G \twoheadrightarrow H$ be an epimorphism with G finitely presented and we have to show that the kernel N of p contains uncountably many subgroups that are normal in G. Since by assumption N has a non-abelian free quotient, this follows from Corollary 4.27. $\square$

5.C.2. *Other examples of extrinsic condensation groups.* Thanks to Lemma 5.2(i), infinitely independently presented (INIP) groups are of extrinsic condensation. This applies to the various examples provided in §4.B. So we content ourselves to show that there are different examples.

We begin with an analogue of Proposition 4.19.

Proposition 5.10. *Let G be a finitely generated group. Suppose that $H_2(G, \mathbf{Z})$ does not satisfy the condition of Proposition 5.6(1). Then G is of extrinsic condensation.*

Proof. If F is a finitely generated free group with an epimorphism $F \twoheadrightarrow G$ with kernel R, then, setting $A = R/[R, F]$, we have a central extension

$$A \rightarrowtail F/[R, F] \twoheadrightarrow G.$$

Arguing exactly as in the proof of Proposition 4.19, A admits $H_2(G, \mathbf{Z})$ as a subgroup and therefore also fails to satisfy the condition of Proposition 5.6(1), and thus A is a condensation element in the set of its own subgroups, which are normal in $F/[R, F]$ because A is central. It follows that G is of extrinsic condensation. $\square$

An application of these ideas yields

Proposition 5.11. *Let Γ be a finitely presented group and Z a central subgroup. Assume that Z is minimax. Then Γ/Z is not infinitely independently presented. If moreover Z admits $\mathbf{Z}(p^\infty)^2$ as a quotient for some prime p, then Γ/Z is of extrinsic condensation.*

Proof. If G were independently presented, since Γ is finitely presented we would get in Z an infinite independent sequence of subgroups. Since Z is minimax, this contradicts Lemma 4.18. The second assertion immediately follows from Proposition 5.6(1). $\square$

Example 5.12. If $n \geq 4$ and $k \geq 2$, and A_n denotes Abels' group with centre $Z \simeq \mathbf{Z}[1/p]$, then the group $G = (A_n/Z)^k$ is of extrinsic condensation but not INIP by Proposition 5.11.

5.D. **Counterexamples to different kinds of condensation.** For finitely presented groups, condensation is equivalent to intrinsic condensation, yielding many examples of non-condensation groups.

- Isolated groups. These include finite groups, finitely presented simple groups, but also many groups having uncountably many normal subgroups. We refer the reader to [CGP07] for a large family of examples.

- Finitely presented groups satisfying max-n; notably, these include finitely presented metabelian groups.

It is more delicate to provide non-condensation infinitely presented groups. The first example is the following.

Example 5.13. In contrast to Example 5.12, if $n \geq 4$, the infinitely presented group $G = A_n/Z$ is not of condensation, as already observed in Corollary 3.6.

Remark 5.14. Example 5.13 provides examples of infinitely presented groups that are not INIP and thus have no minimal presentation (as in Definition 4.4).

In the context of group varieties, Kleiman [Kle83] constructed varieties of groups with no independent defining set of identities. It is however not clear whether the free groups (on $n \geq 2$ generators) in these varieties fail to admit a minimal presentation.

Example 5.15. There exist condensation groups that are
- *extrinsic but not intrinsic.* The group $\mathbf{Z} \wr \mathbf{Z}$ is such an example. It is not of intrinsic condensation because it satisfies max-n while and thus has countably many normal subgroups; on the other hand it is INIP by Example 4.10 and therefore of extrinsic condensation by Lemma 5.2(i).
- *intrinsic but not extrinsic.* The free group F_d is an example for $d \geq 2$. We already mentioned in §5.B.1 that it is of intrinsic condensation. It is finitely presented and thus is clearly not of extrinsic condensation.
- *neither extrinsic nor intrinsic.* The simplest example we are aware of is much more elaborated, and, once more, is based on Abels' group A_4 and its centre $Z \simeq \mathbf{Z}[1/p]$. Namely, $\Gamma = A_4 \times (A_4/Z)$ is such an example.

  It is condensation because $\{0\} \times \mathbf{Z}[1/p]$ is a condensation point in $\mathcal{N}(\mathbf{Z}[1/p]^2)$ (this was shown in [CGP10] but readily follows from the arguments in [Boy56]).

  It is not of extrinsic condensation, since $A_4 \times A_4$ is finitely presentable and the kernel $\{1\} \times Z \simeq \mathbf{Z}[1/p]$ of the projection $A_4 \times A_4 \twoheadrightarrow \Gamma$ has only countably many subgroups.

  Finally, Γ is not of intrinsic condensation because it has countably many normal subgroups, as a consequence of Lemma 3.3.

6. Geometric invariant and metabelian condensation groups

6.A. Splittings.

6.A.1. *The splitting theorem.* Let G be a group and $\pi \colon G \twoheadrightarrow \mathbf{Z}$ an epimorphism. We say that π *splits over a subgroup* $U \subset G$ if there exists an element $t \in G$ and a subgroup $B \subseteq \ker \pi$ which contains both U and $V = tUt^{-1}$, all in such a way that G is canonically isomorphic to the HNN-extension $\mathrm{HNN}(B, \tau \colon U \xrightarrow{\sim} V)$ with vertex group B, edge group U and end point map τ given by conjugation by t. Notice that t plays the rôle of the stable letter. If $U \neq B \neq V$, we say that the splitting is *non-ascending*.

We are interested in the condition that π splits over a *finitely generated* subgroup.

An old result which is relevant in this context is Theorem A of [BS78]; it asserts: *If G is finitely presented then every epimorphism $\pi \colon G \twoheadrightarrow \mathbf{Z}$ splits over a finitely generated subgroup.* We refine this old result to the following structure theorem:

Theorem 6.1. *Let G be a finitely generated group and $\pi \colon G \twoheadrightarrow \mathbf{Z}$ an epimorphism. Then*
  (i) *either π splits over a finitely generated subgroup,*
 (ii) *or G is an inductive limit of a sequence of epimorphisms of finitely generated groups $G_n \twoheadrightarrow G_{n+1}$ ($n \in \mathbf{N}$) with the following features:*
   - *the kernels of the limiting maps $\lambda_n \colon G_n \twoheadrightarrow G$ are free of infinite rank, and*

- *the epimorphisms $\pi \circ \lambda_n : G_n \twoheadrightarrow \mathbf{Z}$ split over a compatible sequence of finitely generated subgroups and have compatible stable letters.*

Remark 6.2. If a *finitely generated* group G splits over a finitely generated subgroup (either as an HNN-extension or as an amalgamated product) then the vertex groups are necessarily finitely generated, too.

6.A.2. *Consequences.* We first develop consequences and applications of Theorem 6.1, which will be proved in §6.A.3.

Corollary 6.3. *Let G be a finitely generated group with an epimorphism $\pi : G \twoheadrightarrow \mathbf{Z}$ that does not split over a finitely generated subgroup. Then G is largely related (see Definition 4.7 and §5.C.1). In particular,*

(a) G is not of type (FP_2) over any non-zero commutative ring K; in particular, G is not finitely presentable.

(b) G is an extrinsic condensation group. $\qquad\qquad\square$

To apply Corollary 6.3, it is useful to clarify whether an epimorphism $G \twoheadrightarrow \mathbf{Z}$ splits over a finitely generated subgroup. Such a splitting can be ascending or not, and the next two propositions deal with each of theses cases.

Definition 6.4. An automorphism $\alpha : N \longrightarrow N$ contracts into a subgroup $B \subset N$ if $\alpha(B) \subset B$ and $N = \bigcup_{n \geq 0} \alpha^{-n}(B)$.

Proposition 6.5. *Consider an epimorphism $\pi : G \twoheadrightarrow \mathbf{Z}$ with kernel N. If $t \in G$, let α_t be the automorphism of N defined by $\alpha_t(g) = tgt^{-1}$. The following statements are equivalent*

(i) π has an ascending splitting over a finitely generated subgroup;

(ii) for some $t \in \pi^{-1}(\{1\})$, either α_t or α_t^{-1} contracts into a finitely generated subgroup of N;

(iii) for every $t \in \pi^{-1}(\{1\})$, either α_t or α_t^{-1} contracts into a finitely generated subgroup of N.

Proof. (i)$\Leftrightarrow$(ii)$\Leftarrow$(iii) is obvious. Suppose (ii) and let us prove (iii). So there exists an element $t_0 \in G$ with $\pi(t_0) = 1$ and a sign $\varepsilon \in \{\pm 1\}$ such that $\alpha_{t_0}^{\varepsilon}$ contracts into a finitely generated subgroup of N. Let us check that for every $t \in G$ with $\pi(t) = 1$, α_t^{ε} contracts into a finitely generated subgroup of N.

By assumption, there exists a finitely generated subgroup $M \subset N$ such that

$$M \subseteq \alpha_{t_0}^{-\varepsilon}(M) \text{ and } \bigcup_{j \geq 0} \alpha_{t_0}^{-\varepsilon j}(M) = N.$$

There exists an element $w \in N$ with $t = t_0 \cdot w$, and hence there exists a non-negative integer k with $w \in M' = \alpha_{t_0}^{-\varepsilon k}(M)$. Then M' is a finitely generated subgroup of N and α_t^{ε} contracts into it. $\qquad\qquad\square$

For the case of a non-ascending splitting, there is no characterization other than obvious paraphrases. The following proposition provides a (non-comprehensive) list of necessary conditions, which by contraposition provide obstructions to the existence of a non-ascending splitting.

Proposition 6.6. *Consider an epimorphism $\pi : G \twoheadrightarrow \mathbf{Z}$ with kernel N, with a non-ascending splitting over a finitely generated subgroup. Then each of the following properties holds.*

(i) N is an iterated amalgam

$$\cdots A_{-2} *_{B_{-2}} A_{-1} *_{B_{-1}} A_0 *_{B_0} A_1 *_{B_1} A_2 \cdots$$

    with all B_i finitely generated and all embeddings proper (if G is finitely generated then all A_i can be chosen finitely generated);

(ii) *N is an amalgam of two infinitely generated groups over a finitely generated subgroup;*

(iii) *(assuming that G is finitely generated) there exists $g \in N$ whose centralizer in N is contained in a finitely generated subgroup of N;*

(iv) *there exists an isometric action of N on a tree with two elements acting as hyperbolic isometries with no common end, with finitely generated edge stabilizers;*

(v) *G has a non-abelian free subgroup;*

(vi) *N is not the direct product of two infinitely generated subgroups.*

Proof. Denote by (*) the property that π has a non-ascending splitting over a finitely generated subgroup. We are going to check (*)$\Rightarrow$(i)$\Rightarrow$(ii)$\Rightarrow$(iii), (ii)$\Rightarrow$(iv)$\Rightarrow$(v), and (iv)$\Rightarrow$(vi).

(*)$\Rightarrow$(i) If π has a non-ascending splitting $\mathrm{HNN}(B, \tau \colon U \xrightarrow{\sim} V)$, then N can be described [Ser77, §1.4] as the amalgamation

$$\cdots B *_U B *_U B *_U B \cdots$$

where all left embeddings $U \rightarrowtail B$ are the inclusion, and all right embeddings are given by τ. So (i) holds. If G is finitely generated, then B is finitely generated as well.

(i)$\Rightarrow$(ii) If N has the form (i), then in particular N is the amalgam

$$(\cdots A_{-2} *_{B_{-2}} A_{-1} *_{B_{-1}} A_0) *_{B_0} (A_1 *_{B_1} A_2 *_{B_2} A_3 \cdots),$$

where both vertex groups are infinitely generated and the edge group B_0 is finitely generated.

(ii)$\Rightarrow$(iii) Suppose that $N = A *_B C$ with B finitely generated and $A \neq B \neq C$. Let N act on its Bass-Serre tree, and let w be some hyperbolic element (e.g. ac for $a \in A \smallsetminus B$ and $c \in C \smallsetminus B$), with axis D. Let H be the centralizer of w. Observe that H preserves the axis D of w. Let H_1 be the kernel of the action of H on D. So H_1 is contained in a conjugate of C, say the finitely generated subgroup gCg^{-1}. Since H/H_1 is a group of isometries of the linear tree, it has at most two generators, so H is also contained in a finitely generated subgroup of N.

(ii)$\Rightarrow$(iv)$\Rightarrow$(v). The implication (ii)$\Rightarrow$(v) is explicitly contained in [Bas76, Thm. 6.1], but both implications are folklore and have, at least in substance been proved many times or in broader contexts. A direct proof of (iv)$\Rightarrow$(v) can be found in [CM87, Lem. 2.6]. Let us justify (ii)$\Rightarrow$(iv). For an amalgam, an easy argument (see for instance [Cor09, Lem. 12 and 14]) shows that the action of an amalgam of two groups over subgroups of index at least 2 and 3 respectively on its Bass-Serre tree does not fix any point in the 1-skeleton, nor any end or pair of ends. By [CM87, Thm. 2.7], there are two elements acting as hyperbolic isometries with no common end.

(iv)$\Rightarrow$(vi) Suppose that $N = N_1 \times N_2$ with N_1, N_2 both infinitely generated. Supposing (iv), we can suppose that the action is minimal (otherwise restrict to the convex hull of all axes of all hyperbolic elements). Since the product of two commuting elliptic isometries is elliptic, either N_1 or N_2, say N_1, contains an element w acting as a hyperbolic isometry. Since w centralizes N_2, the axis D of w is N_2-invariant. If N_2 contains a hyperbolic element, it follows that this axis is D, so by repeating the argument, N_1 in turn stabilizes D, so N stabilizes D, a contradiction. Therefore N_2 acts by elliptic isometries. So it either fixes a point in the 1-skeleton, or fixes a unique point at infinity. In the latter case, this point is stabilized by all of N, a contradiction; in the former case, the set of N_2-fixed points is stable and therefore by minimality is all of the tree, i.e., N_2 acts trivially. So the edge stabilizers contain N_2 and therefore are infinitely generated, a contradiction. $\square$

Example 6.7. We provide here a nontrivial example where Corollary 6.3 applies although N contains non-abelian free subgroups. Fix $d \geq 3$. Consider the ring $A = \mathbf{Z}[x_n : n \in \mathbf{Z}]$ of polynomials in infinitely many variables. Let the group $\mathbf{Z}$ act on A by ring automorphisms so that the generator $1 \in \mathbf{Z}$ sends x_n to x_{n+1}. This ring automorphism induces an automorphism ϕ of the subgroup $\mathrm{EL}_d(A) \subset \mathrm{SL}_d(A)$ generated by elementary matrices, i.e. those matrices $e_{ij}(a)$ with $a \in A$ and $1 \leq i \neq j \leq d$, whose entries differ from the identity matrix only at the (i, j) entry, which is a. (Actually $\mathrm{EL}_d(A) = \mathrm{SL}_d(A)$ by a nontrivial result of Suslin [Sus77], but we do not make use of this fact, except in the statement of Corollary 1.11.)

We are going to check that $\mathrm{EL}_d(A) \rtimes_\phi \mathbf{Z}$ is finitely generated, and that $\pi : \mathrm{EL}_d(A) \rtimes \mathbf{Z} \twoheadrightarrow \mathbf{Z}$ does not split over a finitely generated subgroup. So Corollary 6.3 applies; in particular, $\mathrm{EL}_d(A) \rtimes_\phi \mathbf{Z}$ is an extrinsic condensation group.

Let us first check that $\mathrm{EL}_d(A) \rtimes_\phi \mathbf{Z}$ is finitely generated, or equivalently that $\mathrm{EL}_d(A)$ is finitely generated as a $\langle \phi \rangle$-group. By definition, $\mathrm{EL}_d(A)$ is generated by elementary matrices; since $e_{ij}(a)e_{ij}(b) = e_{ij}(a+b)$, those elementary matrices for which a is a monomial are enough; since $e_{ij}(ab) = [e_{ik}(a), e_{kj}(b)]$ (and $d \geq 3$), those for which $a = x_n$ for some n are enough. This shows that $\mathrm{EL}_d(A)$ is generated, as $\langle \phi \rangle$-group, by $\{e_{ij}(x_0) : 1 \leq i \neq j \leq d\}$. So $\mathrm{EL}_d(A) \rtimes_\phi \mathbf{Z}$ is finitely generated.

Clearly ϕ does not contract into a finitely generated subgroup, as it satisfies the following totally opposite property: for every finitely generated subgroup $M \subset \mathrm{EL}_d(A)$ and every $x \in \mathrm{EL}_d(A) \smallsetminus \{1\}$, the set $\{n \in \mathbf{Z} : \phi^n(x) \in M\}$ is finite.

To check that π does not admit any non-ascending splitting, let us check that (iv) of Proposition 6.6 fails. Indeed, we have $N = \bigcup_{n \geq 0} \mathrm{EL}_d(A_n)$ with $A_n = \mathbf{Z}[x_i : -n \leq i \leq n]$ and it is known [CV96, p. 681] that $\mathrm{EL}_d(A_n)$ has Property (FA) (every action on a tree has a fixed point) for all $d \geq 3$. In particular, every isometric action of $\mathrm{EL}_d(A)$ on a tree is by elliptic isometries, contradicting (iv) of Proposition 6.6.

A variant of the result is obtained by considering $\mathrm{GL}_{\mathbf{Z}}(R)$, the group of infinite matrices with entries in a fixed finitely generated ring R (with unit $1 \neq 0$, possibly non-commutative), indexed by $\mathbf{Z} \times \mathbf{Z}$ and that differ from identity matrix entries for finitely many indices only, and its subgroup $\mathrm{EL}_{\mathbf{Z}}(R)$ generated by elementary matrices. Let ψ be the automorphism of $\mathrm{EL}_{\mathbf{Z}}(R)$ shifting indices (so that $\psi(e_{ij}(a)) = e_{i+1,j+1}(a)$). Using arguments similar to those of the previous construction, $\mathrm{EL}_{\mathbf{Z}}(R) \rtimes_\psi \mathbf{Z}$ is finitely generated, $\psi^{\pm 1}$ does not contract into a finitely generated subgroup. Besides, $\mathrm{EL}_{\mathbf{Z}}(R)$ fails to satisfy (iii) of Proposition 6.6. So $\mathrm{EL}_{\mathbf{Z}}(R) \rtimes_\psi \mathbf{Z}$ is an extrinsic condensation finitely generated group. If the K-theory abelian group $K_1(R) \simeq \mathrm{GL}_{\mathbf{Z}}(R)/\mathrm{EL}_Z(R)$ is finitely generated, then the argument also works for $\mathrm{GL}_{\mathbf{Z}}(R) \rtimes_\psi \mathbf{Z}$.

6.A.3. *Proof of Theorem 6.1.* Choose an element $t \in G$ with $\pi(t) = 1$. Since G is finitely generated, there exists a finite subset S in $N = \ker \pi$ such that $\bigcup_{n \in \mathbf{Z}} t^n S t^{-n}$ generates N.

For $n \geq 0$, let B_n be the subgroup generated by the finite set $\bigcup_{0 \leq j \leq n} t^j S t^{-j}$; for $n \geq 1$, set $U_n = B_{n-1}$ and $V_n = t U_n t^{-1}$; note that both U_n and V_n are contained in B_n. Define next G_n to be the HNN-extension

$$\mathrm{HNN}(B_n, t_n \mid \tau_n : U_n \xrightarrow{\sim} V_n),$$

the isomorphism τ_n being given by conjugation by t in G. Then there exist unique epimorphisms $G_n \twoheadrightarrow G_{n+1} \twoheadrightarrow G$ that extend the inclusions $B_n \hookrightarrow B_{n+1} \hookrightarrow N$ and map t_n to t_{n+1} and t_{n+1} to t. These epimorphisms are clearly compatible with the HNN-structure of the groups involved; in particular, there is a natural surjective homomorphism $\phi : \varinjlim G_n \twoheadrightarrow G$.

Let W_n denote the kernel of the limiting map $\lambda_n\colon G_n \twoheadrightarrow G$ and consider the action of G_n on its Bass-Serre tree T. Since λ_n is injective on the vertex group B_n, the action of W_n on T is free and so W_n is a free group.

We claim that ϕ is an isomorphism. This amounts to showing that every element w in the kernel of $G_0 \twoheadrightarrow G$ lies in the kernel of $G_0 \twoheadrightarrow G_n$ for n large enough. Write $w = t^{n_1}s_1 \dots t^{n_k}s_k$ with $n_i \in \mathbf{Z}$ and $s_i \in S^{\pm 1}$. Clearly $\sum n_i = 0$, so w can be rewritten as $\prod_{i=1}^{k} t^{m_i}s_i t^{-m_i}$ for integers $m_i = n_1 + \dots + n_i$. After conjugation by some power of t, we can suppose that all m_i are positive; let m be the maximum of the m_i. Then, in G_m, w belongs to V_m, which is sent one-to-one to G. Therefore $w = 1$ in G_m.

Suppose now that for some n the kernel W_n is generated, as a normal subgroup, by a finite subset $\mathcal{F}$. Since G is the inductive limit of the G_k there exists then an index $m \geq n$ such that the canonical image of $\mathcal{F}$ under the canonical epimorphism $G_n \twoheadrightarrow G_m$ lies in the trivial subgroup $\{1\}$ of G_m. The canonical map $G_n \twoheadrightarrow G_m$ induces therefore an epimorphism $G_n/W_n \twoheadrightarrow G_m$. It follows that the composition $G_n/W_n \twoheadrightarrow G_m \twoheadrightarrow G$ is an isomorphism, whence $\lambda_m\colon G_m \twoheadrightarrow G$ is bijective and so statement (i) holds. If, on the other hand, each kernel W_n is infinitely generated as a normal subgroup, assertion (ii) is satisfied.

6.B. **The geometric invariant** $\Sigma(G)$. The question whether and how an epimorphism $\pi\colon G \twoheadrightarrow \mathbf{Z}$ splits over a finitely generated subgroup can profitably be investigated with the help of the invariant $\Sigma(G)$ introduced in [BNS87].

Definition 6.8. $\Sigma(G)$ is the conical subset of $\mathrm{Hom}(G, \mathbf{R})$ consisting of all *non-zero* homomorphisms $\chi\colon G \to \mathbf{R}$ with the property that the commutator subgroup G' is finitely generated as P-group for some finitely generated submonoid P of $G_\chi = \{g \in G \mid \chi(g) \geq 0\}$.

The complement of $\Sigma(G)$ in $\mathrm{Hom}(G, \mathbf{R})$ will be denoted by $\Sigma^c(G)$.

In [BNS87], the invariant is denoted by $\Sigma_{G'}(G)$ and is defined as the projection of the above to the unit sphere, or equivalently the set of rays of the vector space $\mathrm{Hom}(G, \mathbf{R})$. Since $\Sigma(G)$ is stable by multiplication by positive real numbers, these are obviously equivalent data.

It is an important non-trivial fact that $\Sigma(G)$ is an open subset of $\mathrm{Hom}(G, \mathbf{R})$.

There are various equivalent definitions of $\Sigma(G)$ and there exists a restatement of the condition $\chi \in \Sigma(G)$ in the case where $\chi(G) = \mathbf{Z}$ that is particularly concise. Indeed, the equivalence of (i) and (iii) in [BNS87, Prop. 4.3] yields

Proposition 6.9. *Let G be a finitely generated group, $\chi\colon G \twoheadrightarrow \mathbf{Z}$ an epimorphism, and $t \in G$ with $\chi(t) = 1$. Then $\chi \in \Sigma(G)$ if and only if the action of t^{-1} on $N = \ker \chi$ contracts into a finitely generated subgroup.*

6.B.1. *Application to extrinsic condensation groups.* Definition 6.8 implies that $\Sigma(G)$ is invariant under multiplication by positive real numbers. Proposition 6.9 allows us therefore to deduce from Corollary 6.3 the following result:

Corollary 6.10. *Let G be a finitely generated group. If $\Sigma^c(G)$ contains a rational line $\mathbf{R}\chi$ with $\chi = G \twoheadrightarrow \mathbf{Z}$ such that χ does not have a non-ascending splitting, then G is an extrinsic condensation group.*

Example 6.11. Let $\chi_i\colon G_i \twoheadrightarrow \mathbf{Z}$, $i = 1, 2$ be epimorphisms. Assume that $\chi_i \in \Sigma^c(G_i)$ for $i = 1, 2$. Given two positive integers m_1, m_2, define G as the fibre product

$$G = \{(g_1, g_2) \in G_1 \times G_2) \mid m_1\chi_1(g_1) + m_2\chi_2(g_1) = 0\}.$$

Then G is an extrinsic condensation group, and in particular is infinitely presented. Indeed, if on G we define $\chi(g_1, g_2) = \chi_1(g_1) = -\chi_2(g_2)$, then it follows from the definition that the rational line $\mathbf{R}\chi$ is contained in $\Sigma^c(G)$. On the other hand, we have $\mathrm{Ker}(\chi) = \mathrm{Ker}(\chi_1) \times \mathrm{Ker}(\chi_2)$ and both factors are infinitely generated because $\chi_i \in \Sigma^c(G_i)$, so by Proposition 6.6(vi), we deduce that G does not have a non-ascending splitting over χ. Thus all assumptions of Corollary 6.10 are satisfied.

Remark 6.12. In all cases where $\Sigma^c(G)$ has been determined explicitly the invariant has turned out to be a *polyhedral* cone. And with the exception of certain groups of PL-homeomorphisms (see [BNS87, §8]) all known subsets are, in fact, *rational-polyhedral*. In these cases, the condition "$\Sigma^c(G)$ *contains a rational line*" is, of course, equivalent to "$\Sigma^c(G)$ *contains a line*".

6.C. **Metabelian groups and beyond.** We need the following two major results about the Σ-invariant of the pre-[BNS87]-era: Let G be a finitely generated metabelian group. Then

(1) G is infinitely presented if, and only if, $\Sigma^c(G)$ contains a line (Theorem A(ii) in [BS80]).
(2) $\Sigma^c(G)$ is rational polyhedral (Theorem E in [BG84]).

In view of Corollary 6.10 and the remark following it these two results imply that *a finitely generated metabelian group is of extrinsic condensation if, and only, if it is not finitely presentable.*

Parts of this characterization of metabelian, extrinsic condensation groups are also valid for larger classes of solvable groups. To gain the proper perspective, we begin with a general result. If G is a group, denote by G' its derived group and $\gamma_3(G') = [[G', G'], G']$ the third term of the lower central series of G', so that $G/\gamma_3(G')$ is the largest (2-nilpotent)-by-abelian quotient of G.

Theorem 6.13. *Let G be a finitely generated solvable group. Then the following statements hold:*

(a) if G is finitely presented, so is its metabelianization G/G'';
(b) if G/G'' is finitely presented, the nilpotent-by-abelian quotient $G/\gamma_3(G')$ satisfies max-n and in particular is not of intrinsic condensation;
(c) if G/G'' is not finitely presented, then G is largely related and in particular is of extrinsic condensation.

Proof. Claim (a) is a special case of [BS80, Thm. B]. If G/G'' is finitely presented, then $G/\gamma_3(G')$ satisfies max-n by [BS80, Thm. 5.7], so it has only countably many normal subgroups and thus (b) holds. Assume now that G/G'' does not admit a finite presentation. By properties (1) and (2) stated in the above, the invariant $\Sigma^c(G/G'')$ contains then a rational line. An easy consequence of definition 6.8 is that $\Sigma^c(G/G'') \subset \Sigma^c(G)$, so $\Sigma^c(G)$ contains a rational line as well. Claim (c) now follows from Corollary 6.10. $\square$

Remark 6.14. Let G be a finitely generated (2-nilpotent)-by-abelian group (i.e. with $\gamma_3(G') = \{1\}$). If the metabelianization G/G'' is finitely presented, then G is not of intrinsic condensation by Claim (b) in the preceding proposition. Whether or not it is of (extrinsic) condensation depends, however, on the particular group. To see this consider Abels' group A_4 studied in §3.A. Then $B = A_4/\mathcal{Z}(A_4)$ is (2-nilpotent)-by-abelian, and so is $B \times B$, and their metabelianizations are finitely presented; however, by Examples 5.12 and 5.13, $B \times B$ is of extrinsic condensation but B is not (and is not of condensation either).

Corollary 6.15. *For every finitely generated centre-by-metabelian group G the following statements are equivalent:*

  (i) G is not finitely presented;
  (ii) G/G'' is not finitely presented;
  (iii) $\Sigma^c(G)$ contains a rational line;
  (iv) G is largely related;
  (v) G is of extrinsic condensation;
  (vi) G is of condensation.

Proof. If G/G'' is finitely presented the central subgroup G'' is finitely generated by (b) of Theorem 6.13 and so G itself is finitely presented; implication (i) $\Rightarrow$(ii) therefore holds. Implication (ii) $\Rightarrow$(iii) is a consequence of properties (1) and (2) stated at the beginning of this §6.C and implication (iii) $\Rightarrow$(iv) a consequence of Corollary 6.10. Implication (iv) $\Rightarrow$(v) and (v) $\Rightarrow$(vi) are true for arbitrary G, see Proposition 5.9 and Lemma 5.1(i). For (vi) $\Rightarrow$(i), by contraposition if G is finitely presented, we have by Lemma 5.1(ii) to check that G is not of intrinsic condensation, and this holds by Theorem 6.13(b). $\qquad\square$

6.D. **Thompson's group F.** Recall that a group G is called *finitely discriminable* if $\{1\}$ is isolated in $\mathcal{N}(G)$, or equivalently if there exists a finite family $N_1, \ldots, N_k$ of nontrivial normal subgroups of G such that every nontrivial normal subgroup of G contains one of the N_i. An elementary observation in [CGP07] is that a finitely generated group G is isolated in $\mathcal{G}_m$ (for any marking) if and only if it is both finitely presented and finitely discriminable.

We now turn to the proof of Corollary 1.10. In this paragraph, F denotes Thompson's group of the interval as defined in the introduction. This is a finitely presented group whose derived subgroup $[F, F]$ is an infinite simple group (see [CFP96] for basic properties of F).

Lemma 6.16. *Let N be any normal subgroup of F. Then N is finitely discriminable.*

Proof. We repeatedly use the fact that every nontrivial normal subgroup of F contains the simple group $[F, F]$ (see [CFP96]). This implies in particular that the centralizer of $[F, F]$ is trivial.

The result of the lemma is trivial if $N = 1$. Otherwise, N contains $[F, F]$ (see [CFP96]). If M is a nontrivial normal subgroup of N, since $[F, F]$ has trivial centralizer, it follows that $M \cap [F, F]$ is nontrivial and hence, by simplicity of $[F, F]$, M contains $[F, F]$. This shows that N is finitely discriminable. $\qquad\square$

Let N be a normal subgroup of Thompson's group F such that F/N is infinite cyclic. If in Corollary 1.10, we have $r = 0$, this means that N is equal of $\mathrm{Ker}(\chi_0)$ or $\mathrm{Ker}(\chi_1)$. Otherwise there exists two coprime nonzero integers p, q such that N is equal to the group

$$N_{p,q} = \{f \in F : p\chi_0(f) = q\chi_1(f)\} = \mathrm{Ker}(p\chi_0 - q\chi_1).$$

Lemma 6.17. *The groups $\mathrm{Ker}(\chi_0)$ and $\mathrm{Ker}(\chi_1)$ are infinitely generated, while for any nonzero coprime integers p, q, the group $N_{p,q}$ is finitely generated.*

Proof. Observe that $\mathrm{Ker}(\chi_0)$ is the increasing union of subgroups $\bigcup\{g \in F : g_{|[0,1/n]} = \mathrm{Id}\}$ and $\mathrm{Ker}(\chi_1)$ can be written similarly; thus both groups are infinitely generated.

Now let p, q be nonzero coprime integers. Since $N = N_{p,q}$ acts 2-homogeneously (i.e., transitively on unordered pairs) on $X =]0, 1[\cap \mathbf{Z}[1/2]$ (because its subgroup $[F, F]$ itself acts 2-homogeneously), the stabilizer $N_{1/2}$ of $1/2$ is maximal in N and therefore it is enough to check that $N_{1/2}$ is finitely generated.

Let i_0, i_1 be the following endomorphisms of F: if $f \in F$, $i_0(f)$ (resp. $i_1(f)$) acts on $[0, 1/2]$ (resp. $[1/2, 1]$) as f acts on $[0, 1]$ (i.e. $i_0(f)(t) = f(2t)/2$ for $0 \leq t \leq 1/2$ and $i_1(f)(t) = 1/2 + f(2t-1)/2$ for $1/2 \leq t \leq 1$); while $i_0(f)$ is the identity on $[1/2, 1]$ and $i_1(f)$ is the identity on $[0, 1/2]$.

Define $F_k = \{g \in F : \chi_0(f) \in k\mathbf{Z}\}$ and similarly $F^k = \{g \in F : \chi_1(f) \in k\mathbf{Z}\}$; these are finite index subgroups of F and are thus finitely generated. Fix $\sigma \in F$ with slope 2 at 0 and 1. Define, for $f \in F_q$, $j_0(f) = i_0(f)i_1(\sigma)^{p\chi_0(f)/q}$. By construction, $j_0(f) \in N_{1/2}$. Similarly, for $f \in F^p$, define $j_1(f) = i_0(\sigma^{q\chi_1(f)/p})i_1(f)$; then $j_1(f) \in N_{1/2}$.

We claim that $N_{1/2}$ is generated by $j_0(F_q) \cup j_1(F^p)$. Indeed, if $g \in N_{1/2}$, then $\chi_0(g) \in q\mathbf{Z}$, so by composition by an element in $j_0(F_q)$ we obtain an element which is the identity on $[0, 1/2]$. In turn, by composition by an element of $j_1(F^p)$, we obtain an element which is the identity on $[1/2, 1]$ and is a certain power $i_0(\sigma)^k$ on $[0, 1/2]$; but since we obtained an element of N, necessarily $k = 0$. This proves the claim and thus $N_{1/2}$, and hence N, are finitely generated. $\square$

Remark 6.18. There is a more conceptual proof of Lemma 6.17, based on the geometric invariant. It uses [BNS87, Thm. B1], which says that if N is a normal subgroup of a finitely generated group G, then N is finitely generated if and only if $\Sigma^c(G) \cap \mathrm{Hom}(G/N, \mathbf{R}) = \{0\}$, where $\mathrm{Hom}(G/N, \mathbf{R})$ denotes the set of homomorphisms $G \to \mathbf{R}$ vanishing on N.

A simple verification [BNS87, Thm. 8.1] shows that $\Sigma^c(F)$ consists of the two half-lines generated by χ_0 and χ_1. This yields the statement of Lemma 6.17.

Lemma 6.19. *If $pq > 0$ then $N_{p,q}$ is isomorphic to an ascending HNN-extension of F and is an isolated (hence finitely presented) group.*

Proof. Set $N = N_{p,q}$ and let $t \in N$ generate $N/[F, F]$, so that $\chi_0(t)$ is positive (hence $\chi_1(t) > 0$ as well). Then there exists a dyadic segment I, contained in $]0, 1[$, such that $t(I) \subset I$ and $\bigcup_{n \geq 0} t^{-n}(I) =]0, 1[$ (if $\alpha > 0$ is a small enough dyadic number, then $[\alpha, 1-\alpha]$ is such a segment). If $F(I)$ is the Thompson group in the interval I, then $tF(I)t^{-1} = F(t(I)) \subset F(I)$ and $\bigcup_{n \geq 0} t^{-n}F(I)t^n = [F, F]$, and thus N is the ascending HNN-extension with stable letter t and vertex group $F(I)$. In particular, N is finitely presented. Since by Lemma 6.16 it is finitely discriminable, it follows that N is isolated. $\square$

Lemma 6.20. *If $pq < 0$ then $\Sigma^c(N_{p,q}) = \mathrm{Hom}(N_{p,q}, \mathbf{R}) = \mathbf{R}$.*

Proof. Set $N = N_{p,q}$. Let us check that $\chi = (\chi_0)_{|N}$ belongs to $\Sigma^c(N)$. We have to check that for any finite subset S of F_χ generating a submonoid M_S, $[F, F]$ is not finitely generated as an M_S-group. There exists $\varepsilon > 0$ such that for every $s \in S$, the element s is linear in the interval $[0, \varepsilon]$ with slope at least one. It follows that if H_n is the set of elements of $[F, F]$ that are equal to 1 in the interval $[0, 2^{-n}\varepsilon]$, then H_n is an M_S-subgroup. Since $[F, F]$ is the increasing union $\bigcup H_n$, it is therefore not finitely generated as an M_S-subgroup.

The same argument shows that $(\chi_1)_{|N}$ belongs to $\Sigma^c(N)$. Since $pq < 0$, $\mathrm{Hom}(N, \mathbf{R})$ is the union of the half-lines generated by $(\chi_0)_{|N}$ and $(\chi_1)_{|N}$, and we are done. $\square$

Remark 6.21. We do not use this fact, but the reader can check that if $pq > 0$ and $N = N_{p,q}$ then $\Sigma^c(N)$ is the half-line generated by $(\chi_0)_{|N}$ (which also contains $(\chi_1)_{|N}$).

Corollary 6.22. *If $pq < 0$ then $N_{p,q}$ is an extrinsic condensation group and in particular is infinitely presented.*

Proof. By Lemma 6.17, $N_{p,q}$ is finitely generated. So Lemma 6.20 together with Corollary 6.10 imply that $N_{p,q}$ is of extrinsic condensation (and thus infinitely presented). $\square$

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